UNKNOTTED CURVES ON GENUS ONE SEIFERT SURFACES OF WHITEHEAD DOUBLES

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ABSTRACT. We consider homologically essential simple closed curves on Seifert surfaces of genus one knots in S^3, and in particular those that are unknotted or slice in S^3. We completely characterize all such curves for most twist knots: they are either positive or negative braid closures; moreover, we determine exactly which of those are unknotted. A surprising consequence of our work is that the figure eight knot admits infinitely many unknotted essential curves up to isotopy on its genus one Seifert surface, and those curves are enumerated by Fibonacci numbers. On the other hand, we prove that many twist knots admit homologically essential curves that cannot be positive or negative braid closures. Indeed, among those curves, we exhibit an example of a slice but not unknotted homologically essential simple closed curve. We further investigate our study of unknotted essential curves for arbitrary Whitehead doubles of non-trivial knots, and obtain that there is a precisely one unknotted essential simple closed curve in the interior of the doubles' standard genus one Seifert surface. As a consequence of all these we obtain many new examples of 3-manifolds that bound contractible 4-manifolds.

1. INTRODUCTION

Suppose $K \subseteq S^3$ is a genus g knot with Seifert Surface Σ_K. Let b be a curve in Σ_K which is *homologically essential*, that is it is not separating Σ_K, and *a simple closed curve*, that is it has one component and does not intersect itself. Furthermore, we will focus on those that are *unknotted* or *slice* in S^3, that is each bounds a disk in S^3 or B^4. In this paper we seek to progress on the following problem:

Problem. *Characterize and, if possible, list all such b's for the pair (K, Σ_K) where K is a genus one knot and Σ_K its Seifert surface.*

Our original motivation for studying this problem comes from the intimate connection between unknotted or slice homologically essential curves on a Seifert surface of a genus one knot and 3-manifolds that bound contractible 4-manifolds. We defer the detailed discussion of this connection to Section 1.2, where we also provide some historical perspective. For now, however, we will focus on getting a hold on the stated problem above for a class of genus one knots, and as we will make clear in the next few results, this problem is already remarkably interesting and fertile on its own.

1.1. **Main Results.** A well studied class of genus one knots is so called twist knot $K = K_t$ which is described by the diagram on the left of Figure 1 (*cf.* [2, Page 182]). We note that with this convention K_{-1} is the right-handed trefoil $T_{2,3}$ and K_1 is the figure eight knot 4_1. We will consider the genus one Seifert surface Σ_K for $K = K_t$ as depicted on the right of Figure 1.

The first main result in this paper is the following.

Theorem 1.1. *Let $t \leq 2$. Then the genus one Seifert surface Σ_K of $K = K_t$ admits infinitely many homologically essential, unknotted curves, if and only if $t = 1$, that is K is the figure eight knot 4_1.*

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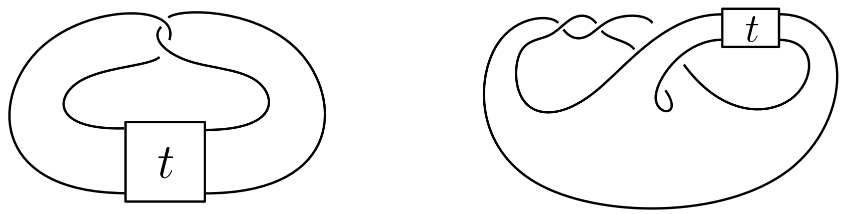

FIGURE 1. On the left is the twist knot K_t where the box contains t full right-handed twists if $t \in \mathbb{Z}_{>0}$, and $|t|$ full left-handed twists if $t \in \mathbb{Z}_{<0}$. On the right is the standard Seifert surface for K_t.

Indeed, we can be more precise and characterize all homologically essential, simple closed curves on Σ_K, from which Theorem 1.1 follows easily. To state this we recall an essential simple closed curve c on Σ_K can be represented (almost uniquely) by a pair of non-negative integers (m, n) where m is the number of times $c = (m, n)$ runs around the left band and n is the number of times it runs around the right band in Σ_K. Moreover, since c is connected, we can assume $\gcd(m, n) = 1$. Finally, to uniquely describe c, we adopt the notation of ∞ *curve* and *loop curve* for a curve c, if the curve has its orientation switches one band to the other and it has the same orientation on both bands, respectively (See Figure 9).

Theorem 1.2. *Let $K = K_t$ be a twist knot and Σ_K its Seifert surface as in Figure 1. Then;*

(1) *For $K = K_t$, $t \leq -1$, we can characterize all homologically essential simple closed curves on Σ_K as the closures of negative braids in Figure 10. In case of the right-handed trefoil $K_{-1} = T_{2,3}$, exactly 6 of these, see Figure 2, are unknotted in S^3. For $t < -1$, exactly 5 of these, see Figure 4, are unknotted in S^3.*
(2) *For $K = K_1 = 4_1$, we can characterize all homologically essential simple closed curves on Σ_K as the closures of braids in Figure 15. A curve on this surface is unknotted in S^3 if and only if it is (1) a trivial curve $(1, 0)$ or $(0, 1)$, (2) an ∞ curve in the form of (F_{i+1}, F_i), or (3) a loop curve in the form of (F_i, F_{i+1}), where F_i represents the i^{th} Fibonacci number, see Figure 3.*

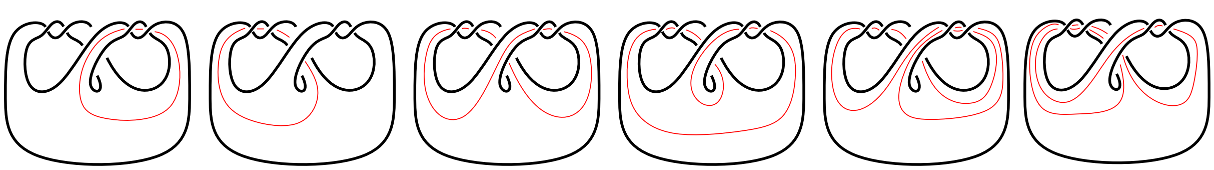

FIGURE 2. It can easily be shown these 6 curves, from left to right $(0, 1), (1, 0), (1, 1)$ ∞, $(1, 1)$ loop, $(1, 2)$ ∞ and $(2, 1)$ ∞, on Σ_K are unknotted in S^3. One can easily check that the other $(1, 2)$ and $(2, 1)$ curves (that is $(1, 2)$ loop and $(2, 1)$ loop curves) both yield the left-handed trefoil $T_{2,-3}$, and hence they are not unknotted in S^3.

For twist knot $K = K_t$ with $t > 1$ the situation is more complicated. Under further hypothesis on the parameters m, n we can obtain results similar to those in Theorem 1.2, and these will be enough to extend the theorem entirely to the case of $K = K_2$, so called Stevedore's knot 6_1 (here we use the KnotInfo database [11] for identifying small knots and their various properties). More precisely we have;

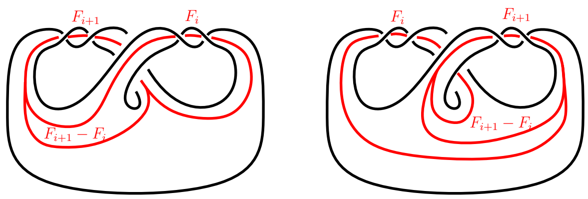

FIGURE 3. The two infinite families of unknotted curves for the figure eight knot in S^3. The letters on parts of our curve or in certain locations stands for the number of strands that particular curve or location. For example, for the (m,n) ∞ curve on the left we will show in Section 3.2 via explicit isotopies how starting with the known unknotted $(1,1)$ ∞ curve we can recursively obtain the following sequence of unknotted curves: $(1,1) \sim (3,2) \sim (8,5) \sim (21,13) \sim (55,34) \sim \cdots$

Theorem 1.3. *Let $K = K_t$ be a twist knot and Σ_K its Seifert surface as in Figure 1. Then;*

(1) *When $t > 1$ and $m < n$, we can characterize all homologically essential simple closed curves on Σ_K as the closures of positive braids in Figure 24(a)(b). Exactly 5 of these, see Figure 4, are unknotted in S^3.*
(2) *When $t > 1$ and $m > n$.*
  (a) *If $m - tn > 0$, then we can characterize all homologically essential simple closed curves on Σ_K as the closures of negative braids in Figure 28 and 31. Exactly 5 of these, see Figure 4, are unknotted in S^3.*
  (b) *If $m - n < n$ and the curve is ∞ curve, then we can characterize all homologically essential simple closed curves on Σ_K as the closures of positive braids Figure 29. Exactly 5 of these, see Figure 4, are unknotted in S^3.*
(3) *For $K = K_2 = 6_1$, we can characterize all homologically essential simple closed curves on Σ_K as the closures of positive or negative braids. Exactly 5 of these, see Figure 4, are unknotted in S^3.*

What Theorem 1.3 cannot cover is the case $t > 2$, $m > n$ and $m - tn < 0$ or when $m - n < n$ and the curve is a loop curve. Indeed in this range *not* every homologically essential curve is a positive or negative braid closure. For example, when $(m,n) = (5,2)$ and $t = 3$ one obtains that the corresponding essential ∞ curve, as a smooth knot in S^3, is the knot $m(5_2)$ (see Figure 34 in Section 5 for a verification of this), and for $(m,n) = (7,3)$ and $t = 3$, the corresponding knot is 10_{132} both of which are known (e.g. via the KnotInfo database [11]) to be not positive braid closures – coincidentally, these knots are not unknotted or slice. Moreover we can explicitly demonstrate, see below, that if one removes the assumption of "∞" from part 2(b) in Theorem 1.3, then the conclusion claimed there fails for certain loop curves when $t > 2$. A natural question is then whether for knot $K = K_t$ with $t > 2$, $m > n$ and $m - tn < 0$ or $m - n < n$ loop curve, there exists unknotted or slice curves on Σ_K other than those listed in Figure 4? A follow up question will be whether there exists slice but not unknotted curves on Σ_K for some $K = K_t$? We can answer the latter question in affirmative as follows:

Theorem 1.4. *Let $K = K_t$ be a twist knot with $t > 2$ and Σ_K its Seifert surface as in Figure 1 and consider the loop curve (m,n) with $m = 3, n = 2$ on Σ_K. Then this curve, as a smooth knot in S^3, is the pretzel knot $P(2t-5,-3,2)$. This knot is never unknotted but it is slice (exactly) when $t = 4$, in which case this pretzel knot is also known as the curious knot 8_{20}.*

Remark 1.5. We note that the choices of m,n values made in Theorem 1.4 are somewhat special in that they yielded an infinite family of pretzel knots, and that it includes a slice but not unknotted curve. Indeed, by using Rudolph's work in [14], we can show (see Proposition 3.8) that the loop curve (m, n) with $m - n = 1, n > 2$ and $t > 4$ on Σ_K, as a smooth knot in S^3, is never slice. The calculation gets quickly complicated once $m - n > 1$, and it stays an open problem if in this range one can find other slice but not unknotted curves.

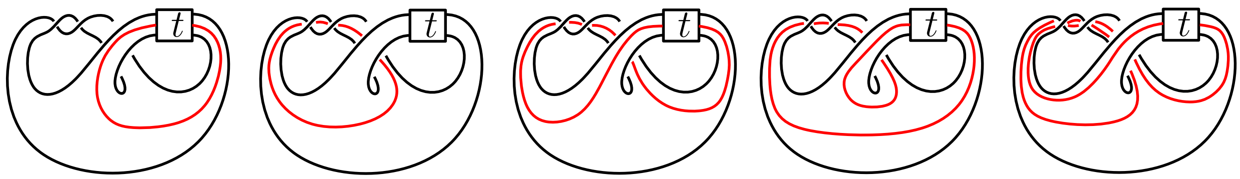

FIGURE 4. These 5 curves, from left to right $(0, 1), (1, 0), (1, 1)\ \infty, (1, 1)$ loop and $(2, 1)\ \infty$, on Σ_K where $K = K_t, t \neq 1$ or -1, are unknotted curves in S^3.

We can further generalize our study of unknotted essential curves on minimal genus Seifert surface of genus one knots for the Whitehead doubles of non-trivial knots. We first introduce some notation. Let P be the twist knot K_t embedded (where $t = 0$ is allowed) in a solid torus $V \subset S^3$, and K denote an arbitrary knot in S^3, we identify a tubular neighborhood of K with V in such a way that the longitude of V is identified with the longitude of K coming from a Seifert surface. The image of P under this identification is a knot, $D^{\pm}(K, t)$, called the positive/negative t–twisted Whitehead double of K. In this situation the knot P is called the pattern for $D^{\pm}(K, t)$ and K is referred to as the companion. Figure 5 depicts the positive -3–twisted Whitehead double of the left-handed trefoil, $D^+(T_{2,-3}, -3)$. If one takes K to be the unknot, then $D^+(K, t)$ is nothing but the twist knot K_t.

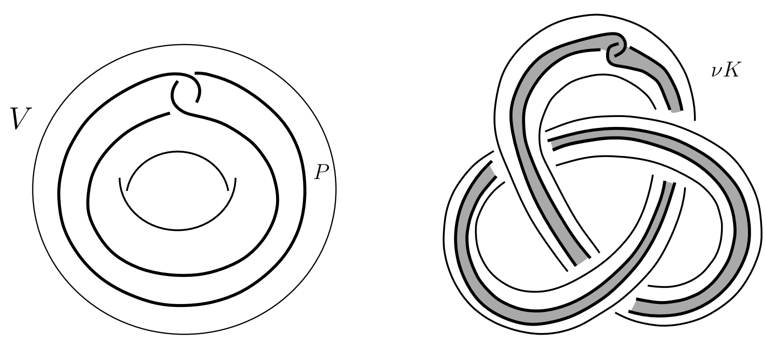

FIGURE 5. On the right is the solid torus $V \subset S^3$ and the pattern twist knot P (which in this case $t = 0$). On the left is the positive -3–twisted Whitehead double of the left-handed trefoil, and its standard genus one Seifert surface.

Theorem 1.6. *Let K denote a non-trivial knot in S^3. Suppose that Σ_K is a standard genus one Seifert surface for the Whitehead double of K. Then there is precisely one unknotted homologically essential, simple closed curves in the interior of Σ_K.*

1.2. **From unknotted curves to contractible 4-manifolds.** The problem of finding unknotted homologically essential curves on a Seifert surface of a genus one knot is interesting on its own, but it is also useful for studying some essential problems in low dimensional topology. We expand on one of these problems a little more. An important and still open question in low dimensional topology asks: *which homology 3-sphere* [1] *bounds a homology 4-ball or contractible 4-manifold* (see [9, Problem 4.2]). This problem can be traced back to the famous Whitney embedding theorem and other important subsequent results due to Hirsch, Wall and Rokhlin [6, 16, 23] in the 1950s. Since then the research towards understanding this problem has stayed active. It has been shown that many infinite families of homology spheres do bound contractible 4-manifolds [1, 7, 20, 24] and at the same time many powerful techniques and homology cobordism invariants, mainly coming from Floer and gauge theories [8, 13, 15] have been used to obtain constraints. See [19] for a detailed recent survey on various constructions and obstructions mentioned above.

In our case, using our main results, we will be able to list some more homology spheres that bound contractible 4-manifolds. This is because of the following theorem of Fickle [7, Theorem 3.1] which was one of the main motivation for the research in this paper.

Theorem 1.7 (Fickle). *Let K be a knot in S^3 which has a genus one Seifert surface F with a primitive element $[b] \in H_1(F)$ such that the curve b is unknotted in S^3. If b has self-linking s, then the homology 3-sphere obtained by $\frac{1}{(s\pm1)}$ Dehn surgery on K bounds a contractible* [2] *4-manifold.*

Theorem 1.7 was generalized (along with a somewhat more accessible reproof of Fickle's theorem) by Etnyre and Tosun in [5, Theorem 1] to genus one knots in the boundary of a homology 4–ball W, and where the assumption on the curve b is relaxed so that b is slice in W. This will be useful, see Corollary 1.9 below, for applying to the slice but not unknotted curve/knot found in Theorem 1.4.

We also want to take the opportunity to highlight an intersting and still open conjecture which is listed in [7, Page 481, Conjecture] and attributed to Fintushel-Stern.

Conjecture 1.8 (Fintushel-Stern). *Let K be a knot in the boundary of a homology 4-ball W which has genus one Seifert surface with a primitive element $[b] \in H_1(F)$ such that b is slice in W. If b has self-linking s, then the homology 3-sphere obtained by $\frac{1}{k(s\pm1)}$, $k \geq 0$, Dehn surgery on K bounds a homology 4-ball*

Corollary 1.9. *Let K_t be a nontrivial twist knot. Then, the homology spheres obtained by*

(1) $\pm\frac{1}{2}$ *Dehn surgery on* $K_1 = 4_1$
(2) $-\frac{1}{2}$ *and* $-\frac{1}{4}$ *Dehn surgeries on* $K_{-1} = T_{2,3}$
(3) $-\frac{1}{2}$ *and* $\frac{1}{t\pm1}$ *and* $\frac{1}{(t-2)\pm1}$ *Dehn surgeries on* K_t, $t \neq \pm1$
(4) $\frac{1}{2}$ *Dehn surgery on* K_4

bound contractible 4-manifolds.

Corollary 1.10. *The homology spheres obtained by $-\frac{1}{2}$ Dehn surgery on $D^+(K,t)$ bounds a contractible 4-manifold.*

Remark 1.11. The 3-manifolds in Corollary 1.9(2) are Brieskorn spheres $\Sigma(2,3,13)$ and $\Sigma(2,3,25)$; they were identified by Casson-Harer and Fickle that they bound contractible 4-manifolds. Also, it was known already that the result of $\frac{1}{2}$ Dehn surgery on the figure eight knot bounds a contractible 4-manifold (see [22, Theorem 18 and Figure 6]), from this we obtain the result in Corollary 1.9(1) as the figure eight knot is an amphichiral knot. The result in Corollary 1.10 also follows from [7, Theorem 3.6].

[1] A homology 3-sphere/4-ball is a closed, oriented, smooth 3-/4-manifold having the integral homology groups of S^3/B^4.

[2] Indeed, this contractible manifold is a *Mazur-type* manifold, namely it is a contractible 4-manifold that has a single handle of each index 0, 1 and 2 where the 2-handle is attached along a knot that links the 1-handle algebraically once. This condition yields a trivial fundamental group.

Remark 1.12. It is known that the result of $\frac{1}{n}$ Dehn surgery on a slice knot $K \subset S^3$ bounds a contractible 4-manifold. To see this, note that at the 4-manifold level with this surgery operation what we are doing is to remove a neighborhood of the slice disk from B^4 (the boundary at this stage is zero surgery on K) and then attach a 2-handle to a meridian of K with framing $-n$. Now, simple algebraic topology arguments shows that this resulting 4-manifold is contractible.

It is a well known result [2] that a nontrivial twist knot $K = K_t$ is slice if and only if $K = K_2$ (Stevedore's knot 6_1). So, by arguments above we already know that result of $\frac{1}{n}$ surgery on K_2 bounds contractible 4-manifold for any integer n. But interestingly we do not recover this by using Theorem 1.3.

Organization. The paper is organized as follows. In Section 2 we set some basic notations and conventions that will be used throughout the paper. Section 3 contains the proofs of Theorem 1.2, 1.3 and 1.4. Our main goal will be to organize, case by case, essential simple closed curves on genus one Seifert surface Σ_K, through sometimes lengthy isotopies, into explicit positive or negative braid closures. Once this is achieved we use a result due to Cromwell that says the Seifert algorithm applied to the closure of a positive/negative braid closure gives a minimal genus surface. This together with some straightforward calculations will help us to determine the unknotted curves exactly. But sometimes it will not be obvious or even possible to reduce an essential simple closed curve to a positive or negative braid closure (see Section 3.2, 3.3, 3.4 and Figure 34 in Section 5). Further analyzing these cases will yield interesting phenomenon listed in Theorem 1.3 and 1.4. Section 4 contains the proof of Theorem 1.6. Finally Section 5 contains the proofs of Corollary 1.9, 1.10 and some final remarks.

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2. Preliminaries

In this section, we set some notation and make preparations for the proofs in the next three sections. In Figure 6 we record some basic isotopies/conventions that will be repeatedly used during proofs. Most of these are evident but for the reader's convenience we explain how the move in part (a) and (f) works in Figure 7 and 8. We remind the reader that letters on parts of our curve, as in part (e) of the figure, or in certain location is to denote the number of strands that particular curve has.

Recall also an essential, simple closed curve on Σ_K can be represented by a pair of non-negative integers (m, n) where m is the number of times it runs around the left band and n is the number of times it runs around the right band in Σ_K, and since we are dealing with connected curves we must have that m, n are relatively prime.

We have two cases: $m > n$ or $n > m$. For an (m, n) curve with $m > n$, after the m strands pass under the n strands on the Seifert surface, it can be split into two sets of strands. For this case, assume that the top set is made of n strands. They must connect to the n strands going over the right band, leaving the other set to be made of $m - n$ strands. Now, we can split the other side of

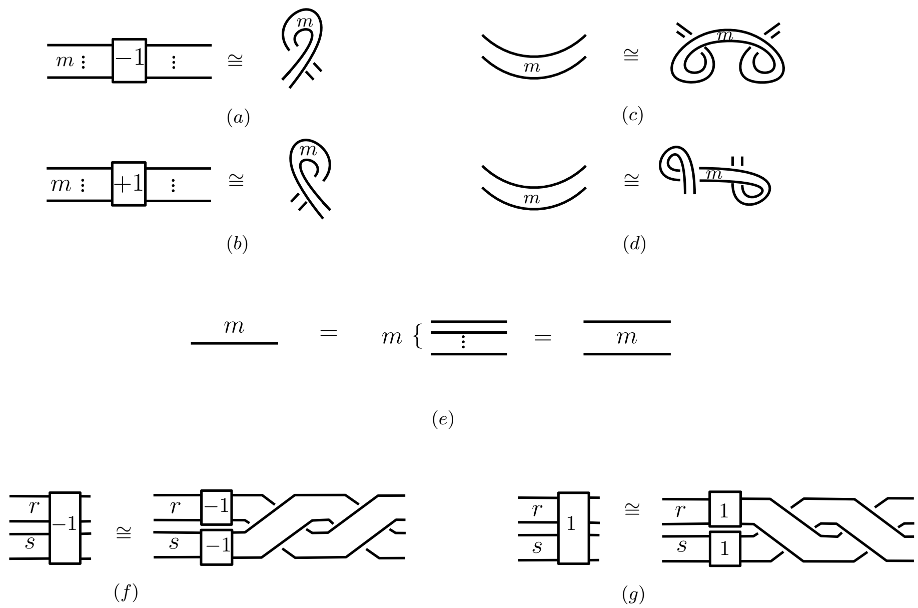

FIGURE 6. Various isotopies.

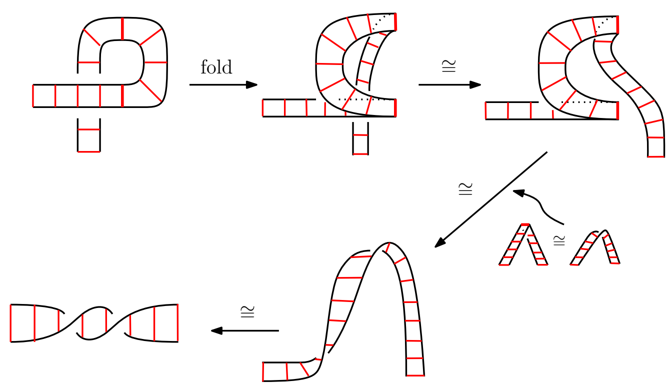

FIGURE 7. Diagrammatic proof of move in Figure 6(a). As indicated the passage from the top right figure to the bottom right is via the "smoothing" a creased edge.

the set of m strands into two sections. The $m - n$ strands on the right can only go to the bottom of these two sections, because otherwise the curve would have to intersect itself on the surface. This curve is notated an (m, n) ∞ curve. See Figure 9(a). The other possibility for an (m, n) curve with $m > n$, has n strands in the bottom set instead, which loop around to connect with the n strands going over the right band. This leaves the other to have $m - n$ strands. We can split the other side of the set of m strands into two sections. The $m - n$ strands on the right can only go to the top of these two sections, because again otherwise the curve would have to intersect itself on the surface. The remaining subsection must be made of n strands and connect to the n strands going over the right band. This curve is notated as an (m, n) loop curve. See Figure 9(b). The case of (m, n) curve with $n > m$ is similar. See Figure 9(c) & (d).

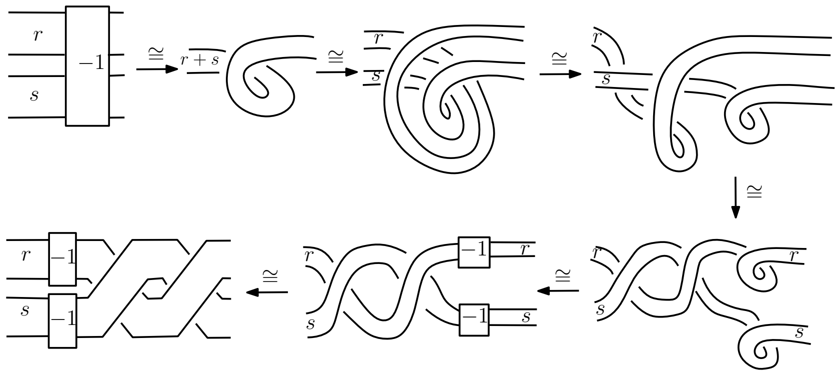

FIGURE 8. Diagrammatic proof of move in Figure 6(f).

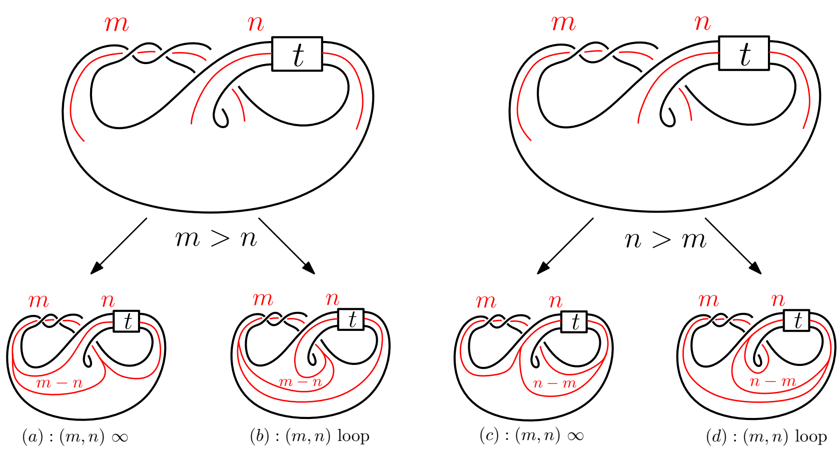

FIGURE 9. Possibilities for an essential, simple closed curve (m, n) on Σ_K.

3. TWIST KNOTS

In this section we provide the proofs of Theorem 1.2, 1.3 and 1.4. We do this in four parts. Section 3.1 and 3.2 contains all technical details of Theorem 1.2, Section 3.3 contains details of Theorem 1.3 and Section 3.4 contains Theorem 1.4 .

3.1. **Twist knot with** $t < 0$**.** In this section we consider twist knot $K = K_t$, $t \leq -1$. This in particular includes the right-handed trefoil K_{-1}.

Proposition 3.1. *All essential, simple closed curves on* Σ_K *can be characterized as the closure of one of the negative braids in Figure 10.*

Proof. It suffices to show all possible curves for an arbitrary m and n such that $gcd(m, n) = 1$ are the closures of either braid in Figure 10. As mentioned earlier we will deal with cases where both $m, n \geq 1$ since cases involving 0 are trivial. There are four cases to consider. The arguments for each of these will be quite similar, and so we will explain the first case in detail and refer to to the rather self-explanatory drawings/figures for the remaining cases.

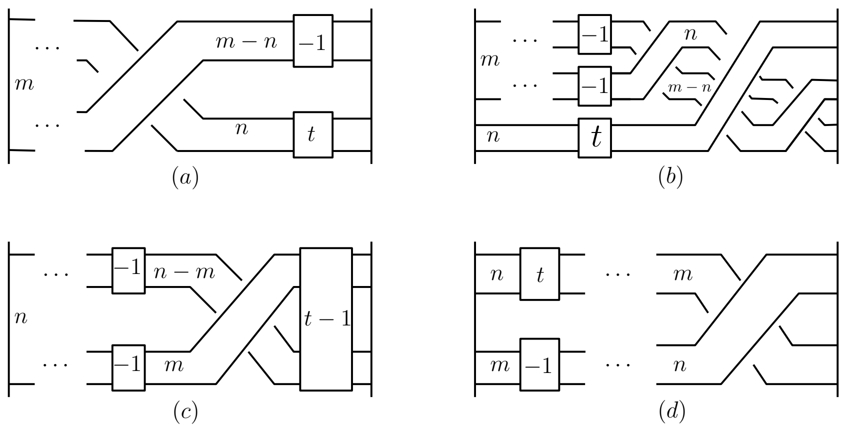

FIGURE 10.

Case 1: (m, n) ∞ *curve with* $m > n > 0$. This case is explained in Figure 11. The picture on top left is the (m, n) curve we are interested. The next picture to its right is the (m, n) curve where we ignore the surface it sits on and use the convention from Figure 6(e). The next picture is an isotopy where we push the split between n strands and $m - n$ strands along the dotted blue arc. The next picture is obtained by simple isotopy. The passage from the top right picture to the bottom right is via Figure 6(c). The passage from the bottom right to the figure on its left is obtained by pushing $m - n$ strands around along the green arc. The goal here is to put the curve in a braid closure position. Finally, by applying simple isotopies and Figure 6(a) repeatedly we replace all the loops with full negative twists. Note that we moved the full negative twist on $m - n$ strands clockwise fashion around to bring it in the bottom of the figure. This gives the picture on the bottom left which is the closure of the negative braid depicted in Figure 10(a).

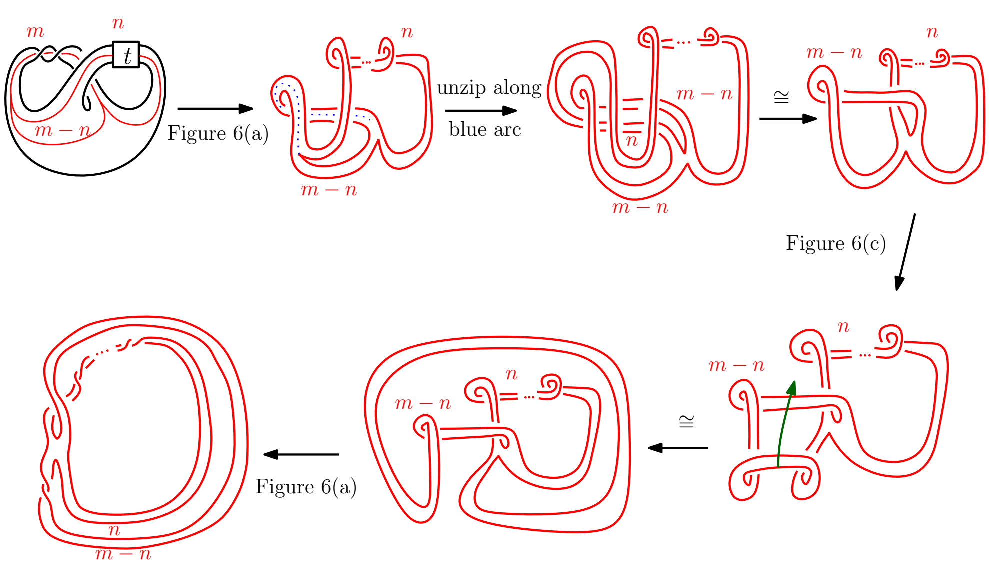

FIGURE 11.

Case 2: (m, n) loop curve with $m > n > 0$. By series isotopies, as indicated in Figure 12, the (m, n) curve in this case can be simplified to the knot depicted on the right of Figure 12, which is the closure of negative braid in Figure 10(b).

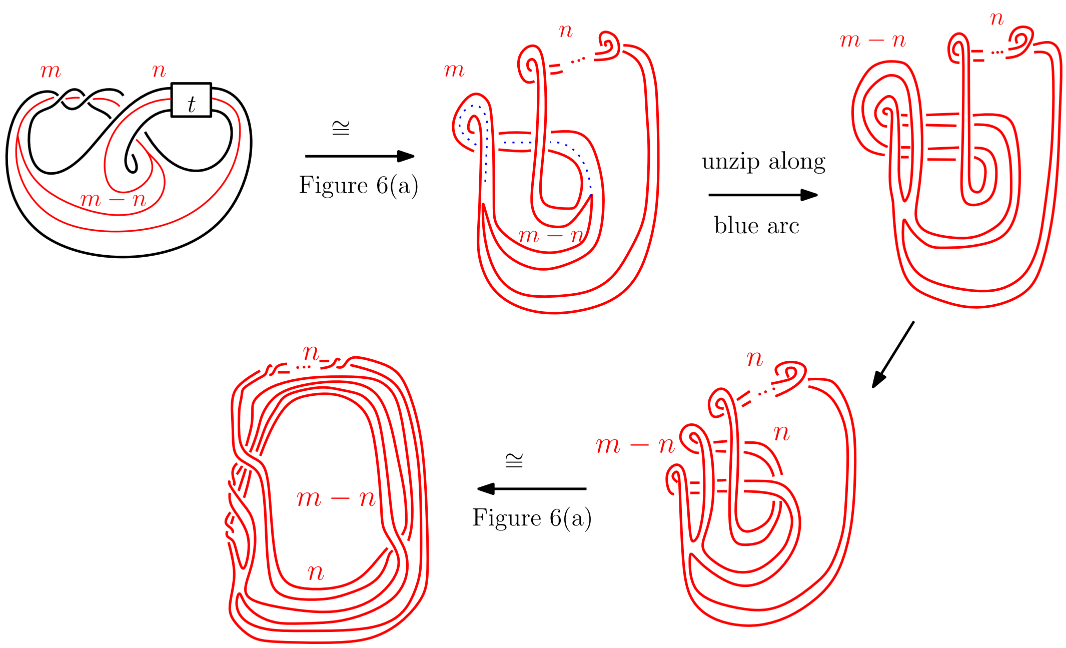

FIGURE 12.

Case 3: (m, n) ∞ curve with $n > m > 0$. By series isotopies, as indicated in Figure 13, the (m, n) curve in this case can be simplified to the knot depicted on the bottom left of Figure 13, which is the closure of negative braid in Figure 10(c).

Case 4: (m, n) loop curve with $n > m > 0$. By series isotopies, as indicated in Figure 14, the (m, n) curve in this case can be simplified to the knot depicted on the right of Figure 14, which is the closure of negative braid in Figure 10(d).

□

Next, we determine which of those curves in Proposition 3.1 are unknotted. It is a classic result due to Cromwell [4] (see also [21, Corollary 4.2]) that the Seifert algorithm applied to the closure of a positive braid gives a minimal genus surface.

Proposition 3.2. *Let β be a braid as in Figure 10 and $K = \hat{\beta}$ be its closure. Let $s(K)$ be the number of Seifert circles and $l(K)$ be the number of crossings in each braid diagram. Then $(s(K), l(K))$ equals to;*

$$\begin{cases} (m, |t|n(n-1) + (m-n)(m-n-1) + n(m-n)) & \beta \text{ as in Fig 10(a)} \\ (m+n, (|t|+1)n(n-1) + (m-n)(m-n-1) + nm + 2n(m-n)) & \beta \text{ as in Fig 10(b)} \\ (n, (|t-1|)n(n-1) + (n-m)(n-m-1) + m(m-1) + m(n-m)) & \beta \text{ as in Fig 10(c)} \\ (m+n, |t|n(n-1) + m(m-1) + nm) & \beta \text{ as in Fig 10(d)} \end{cases}$$

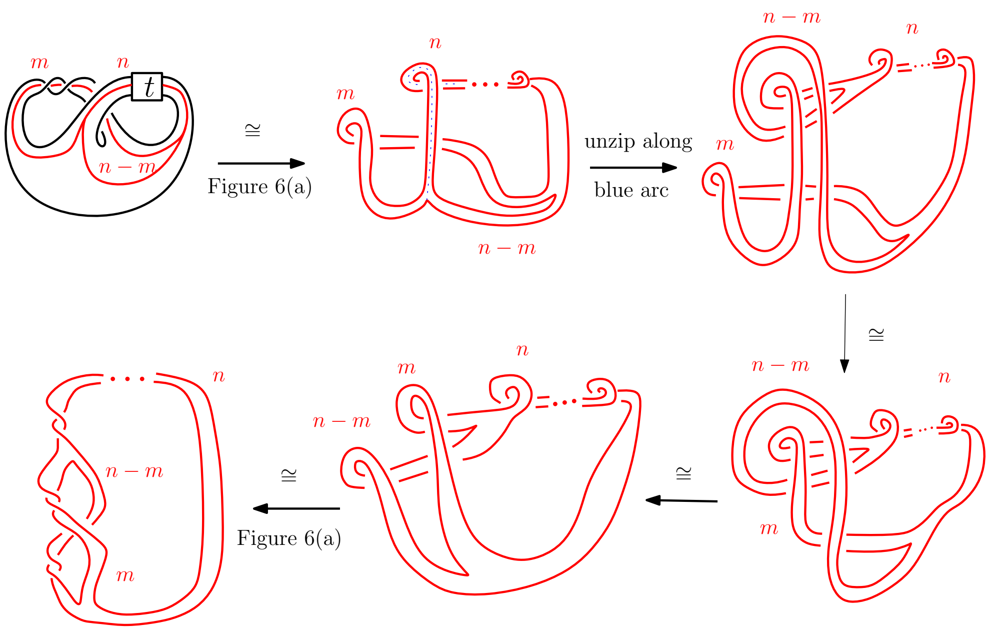

FIGURE 13.

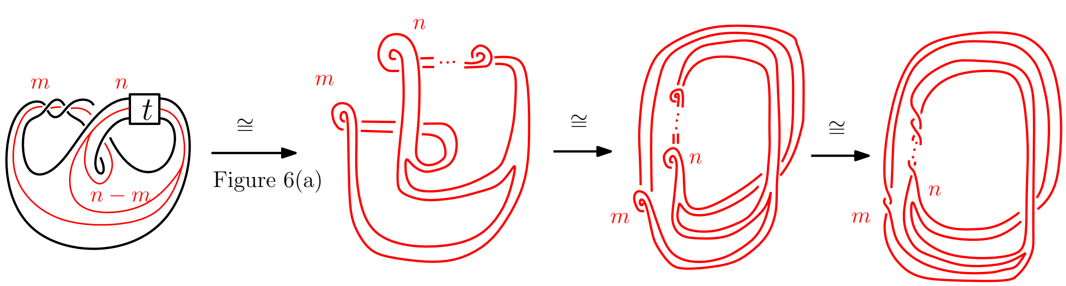

FIGURE 14.

Proof. Consider the braid β as in Figure 10(a). Clearly, it has m Seifert circles as β has m strands. Next, we will analyze the three locations in which crossings occur. First, the t negative full twists on n strands. Since each strand crosses over the other $n-1$ strands, we obtain $|t|n(n-1)$ crossings. Second, the negative full twist on $m-n$ strands produces additional $(m-n)(m-n-1)$ crossings. Lastly, notice the part of β where $m-n$ strands overpass the other n strands, and so for each strand in $m-n$ strands we obtain an additional n crossings. Hence for $K=\hat{\beta}$ we calculate:

$$l(\hat{\beta}) = |t|n(n-1) + (m-n)(m-n-1) + n(m-n).$$

The calculations for the other cases are similar. □

We can now prove the first part of Theorem 1.2.

Proof of Theorem 1.2, part(a). Proposition 3.1 proves the first half of our theorem. To determine there are exactly six unknotted curves when $t = -1$ and five when $t < -1$, let B be the set containing the six and five unknotted curves as in Figure 2 and 4, respectively. It suffices to show an essential, simple closed curve c on Σ_K where $c \notin B$, cannot be unknotted in S^3. We know by Proposition 3.1, c is the closure of one of the braids in Figure 10 in S^3, where $m, n \geq 1,\ gcd(m, n) = 1$. We show, case by case, that the Seifert surface obtained via the Seifert algorithm for curves $c \notin B$ in each case has positive genus, and hence it cannot be unknotted.

- Let $c = (m, n)$ be the closure of the negative braid as in Figure 10(a) and Σ_c its Seifert surface obtained by the Seifert algorithm. There are m Seifert circles and by Proposition 3.2
$$l(c) = |t|n(n-1) + (m-n)(m-n-1) + n(m-n).$$
  Hence,
$$g(\Sigma_c) = \frac{1+l-s}{2} = \frac{m(m-n-2) + n(|t|(n-1)+1) + 1}{2}.$$
  If $m = n+1$, then we get $g(\Sigma_c) = \frac{|t|n(n-1)}{2}$ which is positive as long as $n > 1$–note that when $c = (2, 1)$ we indeed get an unknotted curve. If $m > n+1$, then $g(\Sigma_c) \geq \frac{n(|t|(n-1)+1)+1}{2} > 0$ as long as $n > 0$. So, $c \notin B$ is not an unknotted curve as long as $m > n \geq 1$.
- Let $c = (m, n)$ be the closure of the negative braid as in Figure 10(b) and Σ_c its Seifert surface obtained by the Seifert algorithm. There are $n+m$ Seifert circles and by Proposition 3.2
$$l(c) = (|t|+1)n(n-1) + (m-n)(m-n-1) + nm + 2n(m-n).$$
  Hence,
$$g(\Sigma_c) = \frac{m(m+n-2) + n(|t|(n-1)-1) + 1}{2}.$$
  One can easily see that this quantity is always positive as long as $n \geq 1$. So, $c \notin B$ is not an unknotted curve when $m > n \geq 1$.
- Let $c = (m, n)$ be the closure of the negative braid as in Figure 10(c) and Σ_c its Seifert surface obtained by the Seifert algorithm. There are n Seifert circles and by Proposition 3.2
$$l(c) = (|t|-1)n(n-1) + (n-m)(n-m-1) + m(m-1) + m(n-m).$$
  Hence,
$$g(\Sigma_c) = \frac{n(|t|(n-1) - m - 1) + m^2 + 1}{2}.$$
  This is always positive as long as $m \geq 1$ and $|t| \neq 1$–note that when $c = (1, 2)$ and $|t| = 1$ we indeed get unknotted curve. So, $c \notin B$ is not an unknotted curve when $n > m \geq 1$.
- Let $c = (m, n)$ be the closure of the negative braid as in Figure 10(d) and Σ_c its Seifert surface obtained by the Seifert algorithm. There are $n+m$ Seifert circles and by Proposition 3.2
$$l(c) = |t|n(n-1) + m(m-1) + nm.$$
  Hence,
$$g(\Sigma_c) = \frac{|t|n(n-1) + m(m-2) + n(m-1) + 1}{2}.$$

One can easily see that this quantity is always positive as long as $m \geq 0$. So, $c \notin B$ is not an unknotted curve when $n > m \geq 1$.

This completes the first part of Theorem 1.2.

□

3.2. **Figure eight knot.** The case of figure eight knot is certainly the most interesting one. It is rather surprising, even to the authors, that there exists a genus one knot with infinitely many unknotted curves on its genus one Seifert surface. As we will see understanding homologically essential curves for the figure eight knot will be similar to what we did in the previous section. The key difference develops in Case 2 and 4 below where we show how, under certain conditions, a homologically essential (m, n) ∞ (resp. (m, n) loop) curve can be reduced to the homologically essential $(m - n, 2n - m)$ ∞ (resp. $(2m - n, n - m)$ loop) curve, and how this recursively produces infinitely many distinct homology classes that are represented by the unknot, and we will show that certain Fibonacci numbers can be used to describe these unknotted curves. Finally we will show fort he figure eight knot this is the only way that an unknotted curve can arise. Adapting the notations developed thus far we start characterizing homologically essential simple closed curves on genus one Seifert surface Σ_K of the figure eight knot K.

Proposition 3.3. *All essential, simple closed curves on Σ_K can be characterized as the closure of one of the braids in Figure 15 (note the first and third braids from the left are negative and positive braids, respectively).*

Proof. The curves $(1, 0)$, $(0, 1)$ are clearly unknots. Moreover, because $\gcd(m, n) = 1$, the only curve with $n = m$ is $(1, 1)$ curve, which is also unknot in S^3. For the rest of the arguments below, we will assume $n > m$ or $m > n$. There are four cases to consider:

Case 1: (m, n) loop curve with $m > n > 0$.

This curve can be turned into a negative braid following the process in Figure 16. The reader will observe that the process here is very similar to those in the previous section. We mention that the passage from the middle figure on the top to the one on its right is obtained by pushing the m strands along the green curve till it is clear from a positive loop of n strands. Finally the middle curve on the bottom is our final curve which is the closure of the negative braid to its left.

Case 2: (m, n) ∞ curve with $m > n > 0$. As mentioned at the beginning, this case (and Case 4) are much more involved and interesting (in particular the subcases of Case 2c and 4c). Following the process as in Figure 17, the curve can be isotoped as in the bottom right of that figure, which is the closure of the braid on its left–that is the second braid from the left in Figure 15.

Case 3: (m, n) ∞ curve with $n > m > 0$. This curve can be turned into a positive braid following the process in Figure 18.

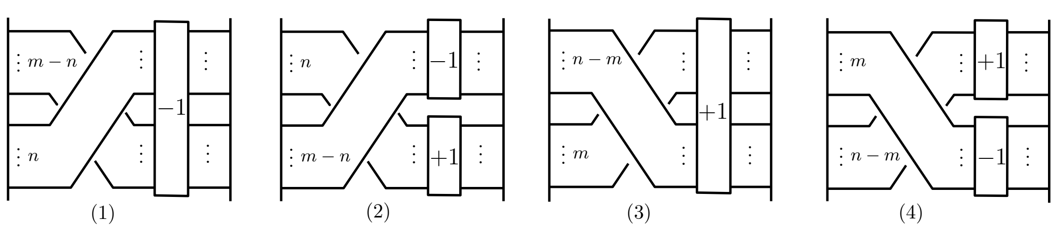

FIGURE 15. Braid representations of curves on Σ_K where K is the figure eight knot. From left to right: (m, n) loop curve with $m > n$; (m, n) ∞ curve with $m > n$; (m, n) ∞ curve with $n > m$; (m, n) loop curve with $n > m$

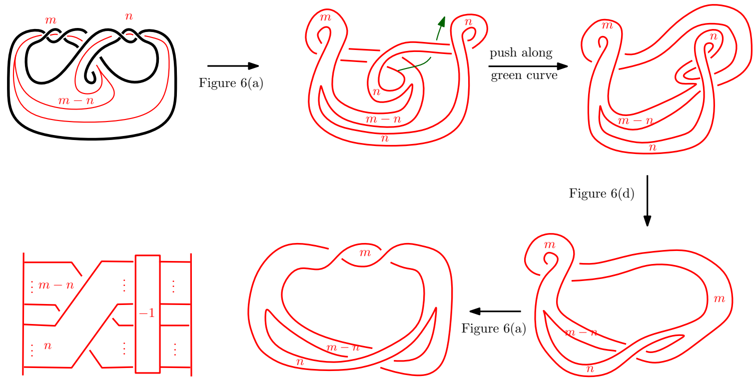

FIGURE 16.

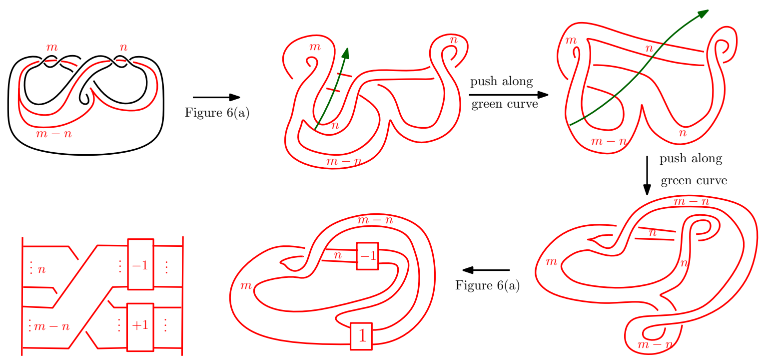

FIGURE 17.

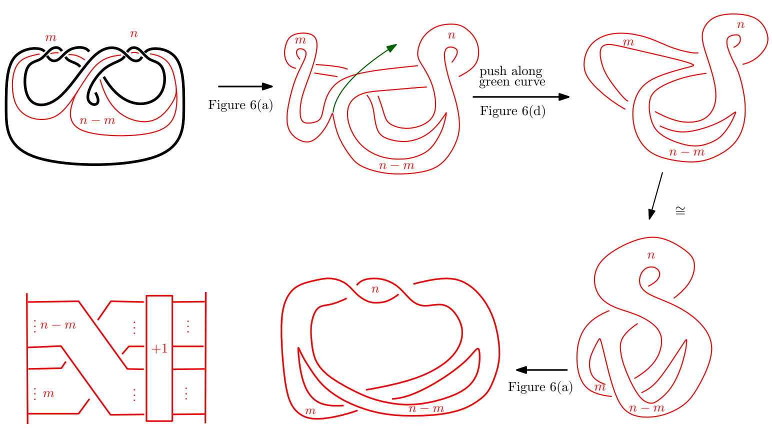

FIGURE 18.

Case 4: (m, n) *loop curve with* $n > m > 0$. This curve can be turned into the closure of a braid following the process in Figure 19.

$\square$

We next determine which of these curves are unknotted:

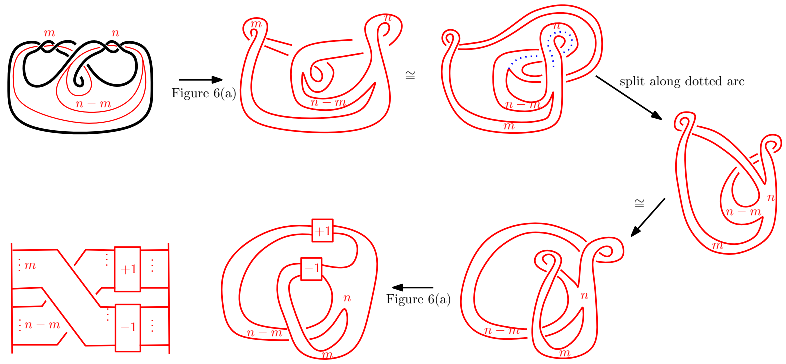

FIGURE 19.

Proposition 3.4. *A homologically essential curve c characterized as in Proposition 3.3 is unknotted if and only if it is (a) a trivial curve $(1,0)$ or $(0,1)$, (b) an ∞ curve in the form of (F_{i+1}, F_i), or (c) a loop curve in the form of (F_i, F_{i+1}).*

Proof. Let c denote one of these homologically essential curve listed in Proposition 3.3. We will analyze the unknottedness of c in four separate cases.

Case 1. Suppose $c = (m,n)$ is the closure of the negative braid in the bottom left of Figure 16. Note the minimal Seifert Surface of c, Σ_c, has $(n)(m-n) + (m)(m-1)$ crossings and m Seifert circles. Hence;

$$g(\Sigma_c) = \frac{n(m-n) + (m-1)^2}{2}.$$

This is a positive integer for all m, n with $m > n$. So c is never unknotted in S^3 as long $m > n > 0$.

Case 2. Suppose c is of the form in the bottom right of Figure 17. Since this curve is not a positive or negative braid closure, we cannot directly use Cromwell's result as in Case 1 or the previous section. There are three subcases to consider.

Case 2a: $m - n = n$. Because m and n are relatively prime integers, we must have that $m = 2, n = 1$, and we can easily see that this $(2,1)$ curve unknotted.

Case 2b: $m - n > n$. This curve can be turned into a negative braid following the process in Figure 20. More precisely, we start, on the top left of that figure, with the curve appearing on the bottom right of Figure 17. We extend the split along the dotted blue arc and isotope m strands to reach the next figure. We note that this splitting can be done as by the assumption we have $m - 2n > 0$. Then using Figure 6(a) and further isotopy we reach the final curve on the bottom right of Figure 20 which is obviously the closure of the negative braid depicted on the bottom left of that picture.

The minimal Seifert Surface coming from this negative braid closure contains $m - n$ circles and $(m-2n)n + (m-n)(m-n-1)$ twists. Hence;

$$g(\Sigma_c) = \frac{(m-2n)n + (m-n)(m-n-2) + 1}{2}.$$

This a positive integer for all integers m, n with $m - n > n$. So, c is not unknotted in S^3.

Case 2c: $m - n < n$. We organize this curve some more. We start, on the top left of Figure 21, with the curve that is appearing on the bottom left of Figure 17. We extend the split along the dotted blue arc and isotope $m - n$ strands to reach the next figure, After some isotopies we reach

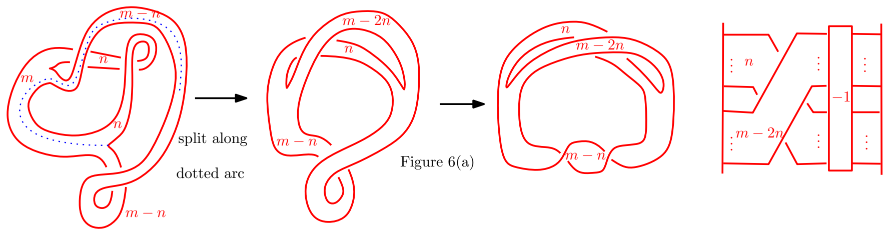

FIGURE 20.

the curve on the bottom left of Figure 21. In other words, this subcase of Case 2c leads to a reduced version of the original picture (top left curve in Figure 17), in the sense that the number of strands over either handle is less than the number of strands in the original picture.

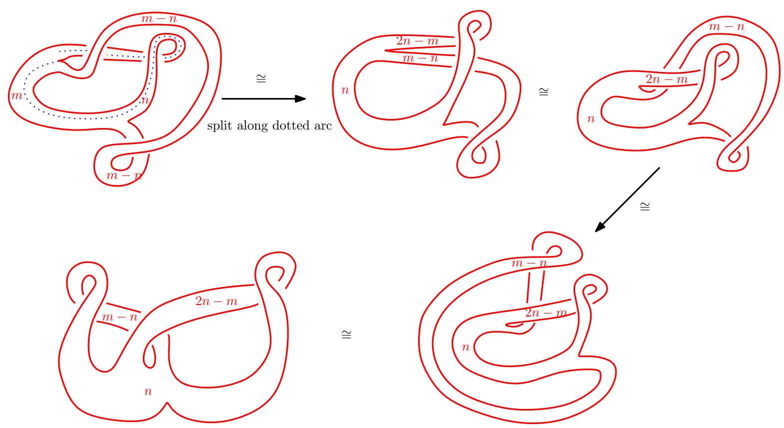

FIGURE 21.

This case can be further subdivided depending on the relationship between $2n-m$ and $m-n$, but this braid (or rather its closure) will turn into a $(m-n, 2n-m)$ ∞ curve when $m-n > 2n-m$:

<u>*Case 2c-i:* $2n-m = m-n$.</u> This simplifies to $3n = 2m$. Because $gcd(m,n) = 1$, this will only occur for $m = 3$ and $n = 2$, and the resulting curve is $(1,1)$ ∞ curve. In other words here we observed that $(3,2)$ curve has been reduced to (1,1) curve.

<u>*Case 2c-ii:* $2n-m > m-n$.</u> This means that we are dealing with a curve under Case 3, and we will see that all curves considered there are positive braid closures.

<u>*Case2c-iii:* $2n-m < m-n$.</u> This means we are back to be under Case 2. So for $m > n > m-n$, the (m,n) ∞ curve is isotopic to the $(m-n, 2n-m)$ ∞ curve. This isotopy series will be notated $(m,n) \sim (m-n, 2n-m)$. Equivalently, there is a series of isotopies such that $(m-n, 2n-m) \sim (m,n)$. If (k,l) denote a curve at one stage of this isotopy, then $(k,l) \sim ((k+l)+k, k+l)$. So, starting with $k = l = 1$, we recursively obtain:

$$(1,1) \sim (3,2) \sim (8,5) \sim (21,13) \sim (55,34) \sim \cdots$$

In a similar fashion, if we start with $k = 2,\ l = 1$ we obtain:

$$(2,1) \sim (5,3) \sim (13,8) \sim (34,21) \sim (89,55) \sim \cdots$$

Notice every curve c above is of the form $c = (F_{i+1}, F_i), i \in \mathbb{Z}_{>0}$ where F_i denotes the i^{th} *Fibonacci number*. We will call these *Fibonacci curves*. We choose $(1,1)$ and $(2,1)$ because they are known unknots. As a result, this relation generates an infinite family of homologically distinct simple closed curves on Σ_K that are unknotted in S^3.

Case 3. Suppose a curve, c, is of the form (3), which is the closure of the positive braid depicted in the bottom left of Figure 18. An argument similar to that applied to Case 1 can be used to show c is never unknotted in S^3.

Case 4. Suppose c is of the form as in the bottom right of Figure 19. Similar to Case 2, there are three subcases to consider.

Case 4a: $m = n - m$. Then $2m = n$. Because $gcd(m, n) = 1$, $m = 1$ and $n = 2$, resulting in unknot.

Case 4b: $n - m > m$. Then $n - 2m > 0$ and following the isotopies in Figure 22, the curve can be changed into the closure of positive braid depicted on the bottom right of that figure.

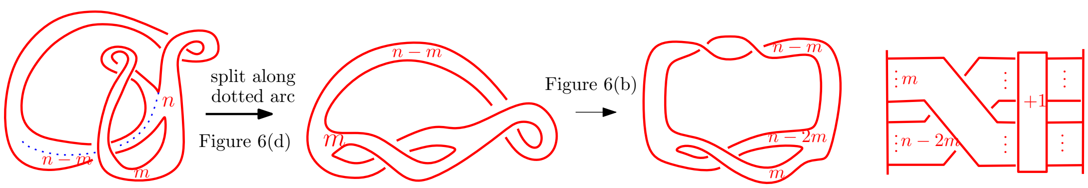

FIGURE 22.

Identical to Case 2b, the curve c in this case is never unknotted in S^3.

Case 4c: $m > n - m$. Then $2m - n > 0$, and we can split the m strands into two: a $n - m$ strands and a $2m - n$ strands.

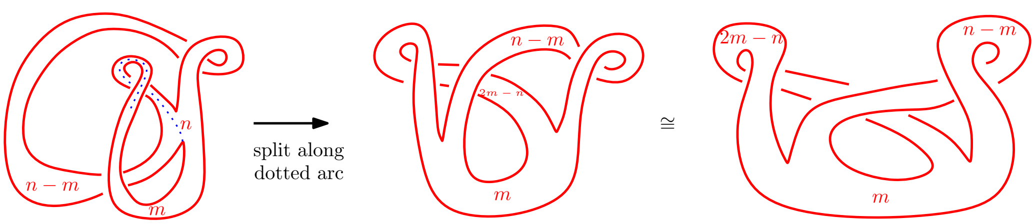

FIGURE 23.

This case can be further subdivided depending on the relationship between $n - m$ and $2m - n$, but this braid will turn into a $(2m - n, n - m)$ loop curve when $n - m > 2m - n$:

<u>*Case 4c-i:* $2m - n = n - m$.</u> This simplifies to $3m = 2n$. Because $gcd(m, n) = 1$, this will only occur for $m = 2$ and $n = 3$, and the resulting curve is a $(1, 1)$ loop curve.

<u>*Case 4c-ii:* $n - m < 2m - n$.</u> This means that we are dealing with a curve under Case 1, and we saw that all curves considered there are negative braid closures.

<u>*Case 4c-iii:* $n - m > 2m - n$.</u> This means that we are back to be under Case 4. So for $n > m > n - m$, an (m, n) loop curve has the following isotopy series: $(m, n) \sim (2m - n, n - m)$. If (k, l) denote a curve at one stage of this isotopy, then the reverse also holds: $(k, l) \sim (k + l, (k + l) + l)$. As a result, much like Case 2c, we can generate two infinite families of unknotted curves in S^3:

$$(1, 1) \sim (2, 3) \sim (5, 8) \sim (13, 21) \sim (34, 55) \sim \cdots \text{ and}$$

$$(1, 2) \sim (3, 5) \sim (8, 13) \sim (21, 34) \sim (55, 89) \sim \cdots$$

Notice every curve c is of the form $c = (F_i, F_{i+1}), i \in \mathbb{Z}_{>0}$. Finally, we show that this is the only way one can get unknotted curves. That is, we claim:

Lemma 3.5. *If a homologically essential curve c on Σ_K for $K = 4_1$ is unknotted, then it must be a Fibonacci curve.*

Proof. From above, it is clear that if our curve c is Fibonacci, then it is unknotted. So it suffices to show if a curve is not Fibonacci then it is not unknotted. We will demonstrate this for loop curves under Case 4. Let c be a loop curve that is not Fibonacci but is unknotted. Since it is unknotted, it fits into either Case 4a or 4c. But the only unknotted curve from Case 4a is $(1, 1)$ curve which is a Fibonacci curve, so c must be under Case 4c. By our isotopy relation, $(m, n) \sim (2m - n, n - m)$. So, the curve can be reduced to a minimal form, say (a, b) where $(a, b) \neq (1, 1)$ and $(a, b) \neq (2, 1)$. We will now analyze this reduced curve (a, b):

- If $a = b$, then $(a, b) = (1, 1)$; a contradiction.
- If $a > b$, then (a, b) is under Case 1; none of those are unknotted.
- If $b - a < a < b$, then (a, b) is still under Case 4c, and not in reduced form; a contradiction.
- If $a < b - a < b$, then (a, b) is under Case 4b; none of those are unknotted.
- If $b - a = a < b$, then $(a, b) = (2, 1)$; a contradiction.

So, it has to be that either $(a, b) \sim (1, 1)$ or $(a, b) \sim (2, 1)$. Hence, it must be that $c = (F_i, F_{i+1})$ for some i. The argument for the case where c is an ∞ curve under Case 2 is identical.

□

□

We end this section with a remark which was observed by the authors at the initial stages of the research and was also communicated to the authors by F. Misev.

Remark 3.6. An alternative and perhaps slightly easier way to see the existence of Fibonacci numbers for unknotted curves for the figure eight knot is as follows: Recall that the figure-eight knot is fibered and its pseudo-Anosov monodromy $\phi : \Sigma \to \Sigma$, where Σ is the genus one Seifert surface, induces a linear map on the first homology $H_1(\Sigma) = \mathbb{Z} \oplus \mathbb{Z}$ described by the matrix:$\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. By applying this matrix repeatedly to the unknotted curves (vectors) $(0, 1)$ and $(1, 0)$ one obtains other unknotted curves that has Fibonacci numbers as their entries exactly as predicted in Proposition 3.4.

We add that this approach cannot capture the full strength of the results about the figure eight knot: namely showing that any unknotted curve as in Lemma 3.5 on genus one Seifert surface of the figure eight knot must be a Fibonacci curve or characterizing all homologically essential curves on the Seifert surface of the figure eight knot as in Proposition 3.3. Moreover our proof technique is by hand and uniform that works for all other twist knots we study in this paper.

3.3. **Twist knot with $t > 1$–Part 1.**

In this section we consider twist knot $K = K_t$, $t \geq 2$, and give the proof of Theorem 1.3.

Proposition 3.7. *All essential, simple closed curves on Σ_K can be characterized as the closure of one of the braids in Figure 24.*

Proof. It suffices to show all possible curves for an arbitrary m and n such that $gcd(m, n) = 1$ are the closures of braids in Figure 24. Here too there are four cases to consider but we will analyze these in slightly different order than in the previous two sections.

Case 1: (m, n) ∞ curve with $n > m > 0$. In this case the curve is the closure of a positive braid, and this is explained in Figure 25 below. More precisely, we start with the curve which is drawn in the top left of the figure, and after a sequence of isotopies this becomes the curve in the bottom

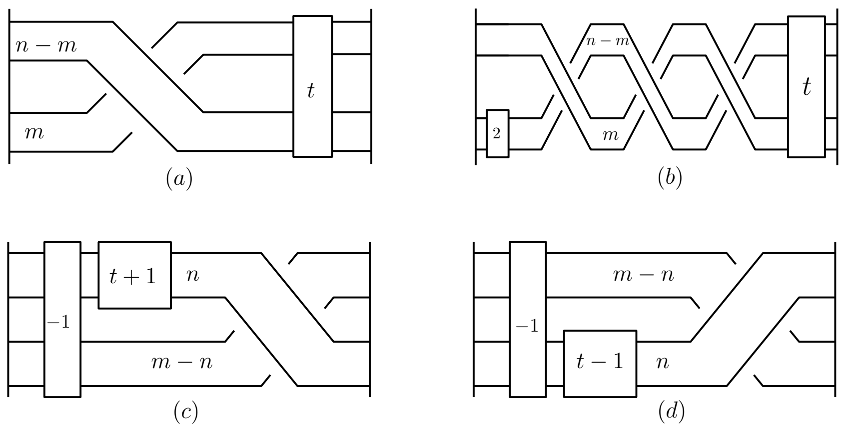

FIGURE 24.

right of the figure which is obviously the closure of the braid in the bottom left of the figure. In particular, when $n > m \geq 1$, none of these curves will be unknotted.

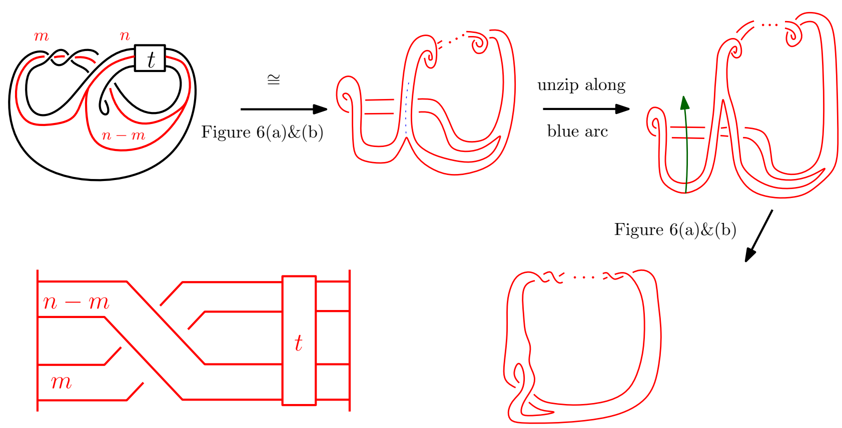

FIGURE 25.

Case 2: (m, n) loop curve with $n > m > 0$. In this case too the the curve is the closure of a positive braid, and this is explained in Figure 26 below. In particular, when $n > m > 1$, none of these curves will be unknotted.

In the remaining two cases we will follow slightly different way of identifying our curves as braid closures. As we will see (which is evident in part (c) and (d) of Proposition 3.7) that the braids will not be positive or negative braids for general and m, n and t values. We will then verify how under the various hypothesis listed in Theorem 1.3 these braids can be reduced to a positive or negative braids.

Case 3: (m, n) ∞ curve with $m > n > 0$. We explain in Figure 27 below how the (m, n) ∞ curve with $m > n > 0$ is the closure of the braid in the bottom left of the figure. This braid is not obviously a positive or negative braid.

Case 3a (m, n) ∞ curve with $m > n > 0$ and $m - tn > 0$. We want to show the braid in the bottom left of Figure 27 under the hypothesis that $m - tn > 0$ can be made a negative braid. We

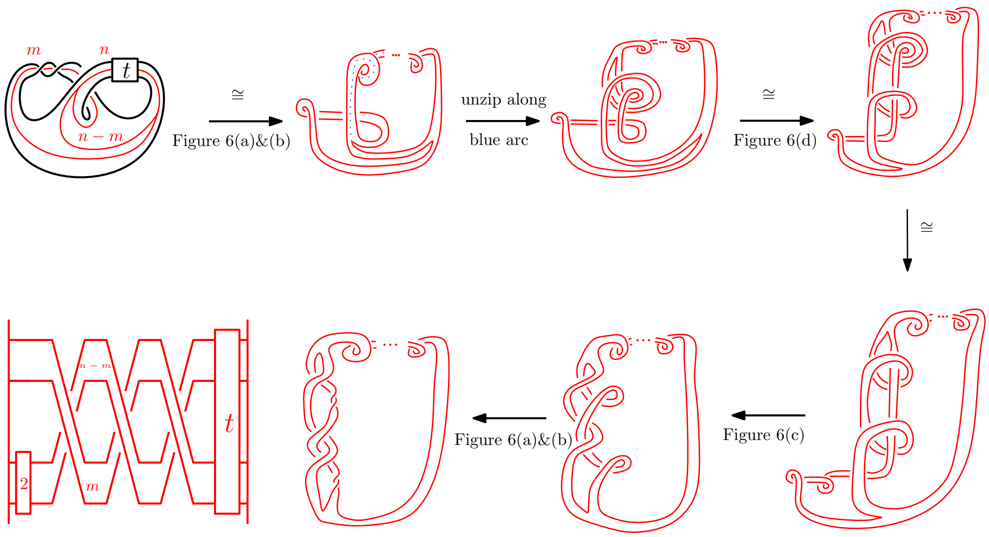

FIGURE 26.

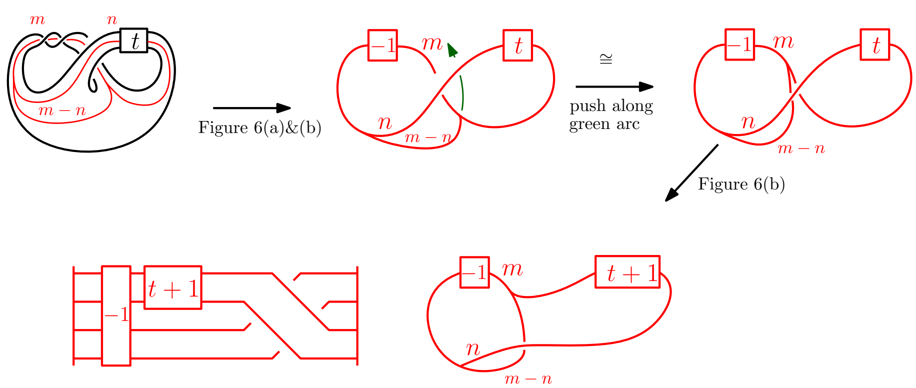

FIGURE 27.

achieve this in Figure 28. More precisely, in part (a) of the figure we see the braid that we are working on. We apply the move in Figure 6(f) and some obvious simplifications to reach the braid in part (d). In part (e) of the figure we re-organize the braid: more precisely, since $m - tn > 0$ and $m - n = m - tn + (t-1)n$, we can split the piece of the braid in part (d) made of $m - n$ strands as the stack of $m - tn$ strands and set of $t - 1$ n strands. We then apply the move in Figure 6(f) repeatedly ($t - 1$ times) to obtain the braid in part (f). We note that the block labeled as "all negative crossings" is not important for our purpose to draw explicitly but we emphasize that each time we apply the move in Figure 6(f) it produces a full left handed twist between an n strands and the rest. Next, sliding -1 full twists one by one from n strands over the block of these negative crossings we reach part (g). After further obvious simplifications and organizations in parts (h)–(j) we reach the braid in part (k) which is a negative braid.

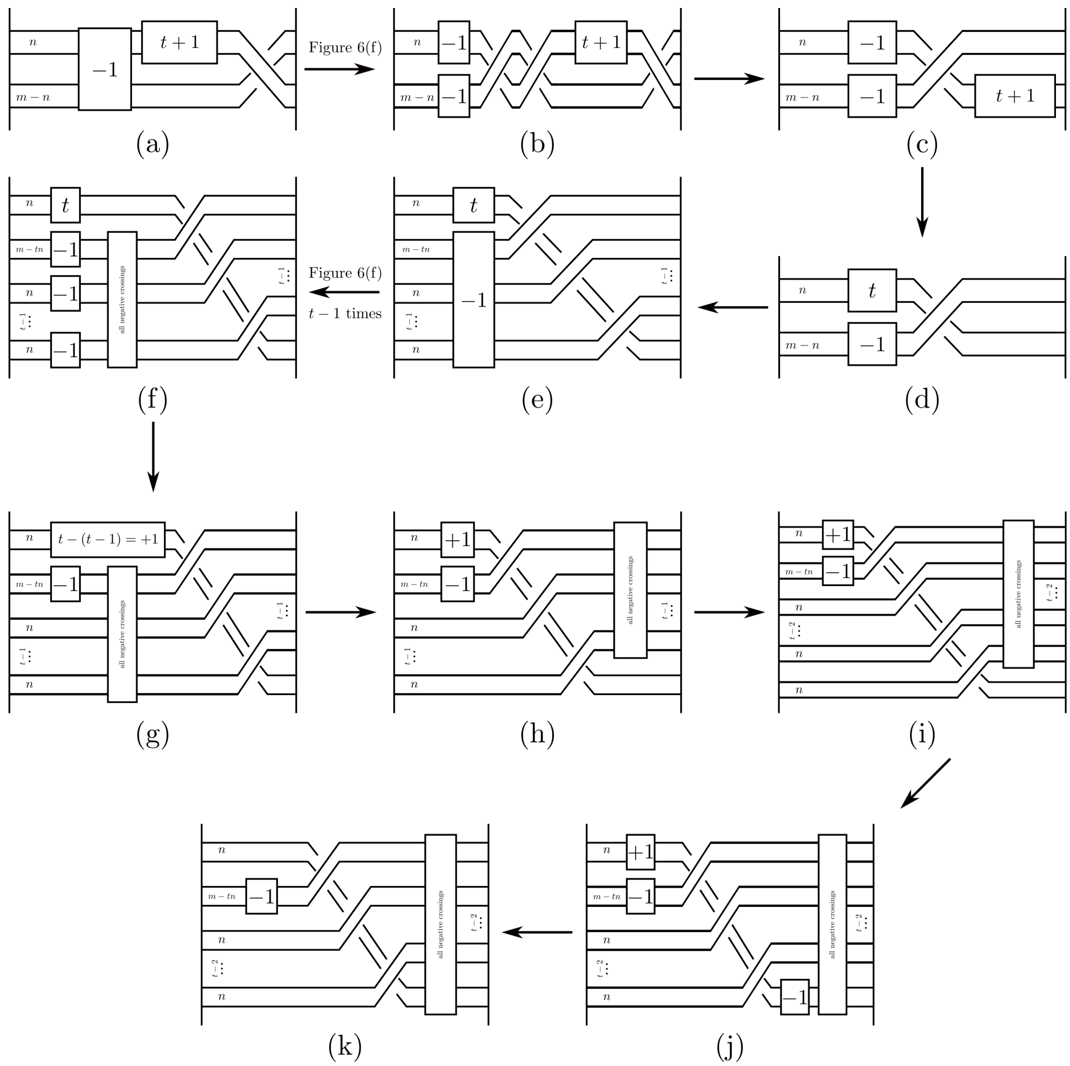

FIGURE 28.

Case 3b (m,n) ∞ curve with $m > n > 0$ and $m-n < n$. We want to show in this case the braid in the bottom left of Figure 27 under the hypothesis that $m-n < n$ can be made a positive braid (regardless of t value). This is achieved in Figures 29.

Case 4: (m,n) loop curve with $m > n > 0$. The arguments for this case are identical Case 3 and 3a above. The (m,n) loop curve with $m > n > 0$ is the closure of the braid that is drawn in the bottom left of Figure 30.

Case 4a (m,n) loop curve with $m > n > 0$ and $m - tn > 0$. We show the braid, which the (m,n) ∞ curve with $m > n > 0$ is closure of, can be made a negative braid under the hypothesis $m - tn > 0$. This follows very similar steps as in Case 3a which is explained through a series drawings in Figure 31.

Case 4b (m,n) loop curve with $m > n > 0$ and $m-n < n$. Finally, we consider the (m,n) loop curve with $m > n > 0$ and $m-n < n$. Interestingly, this curve for $t > 2$ does not have to the

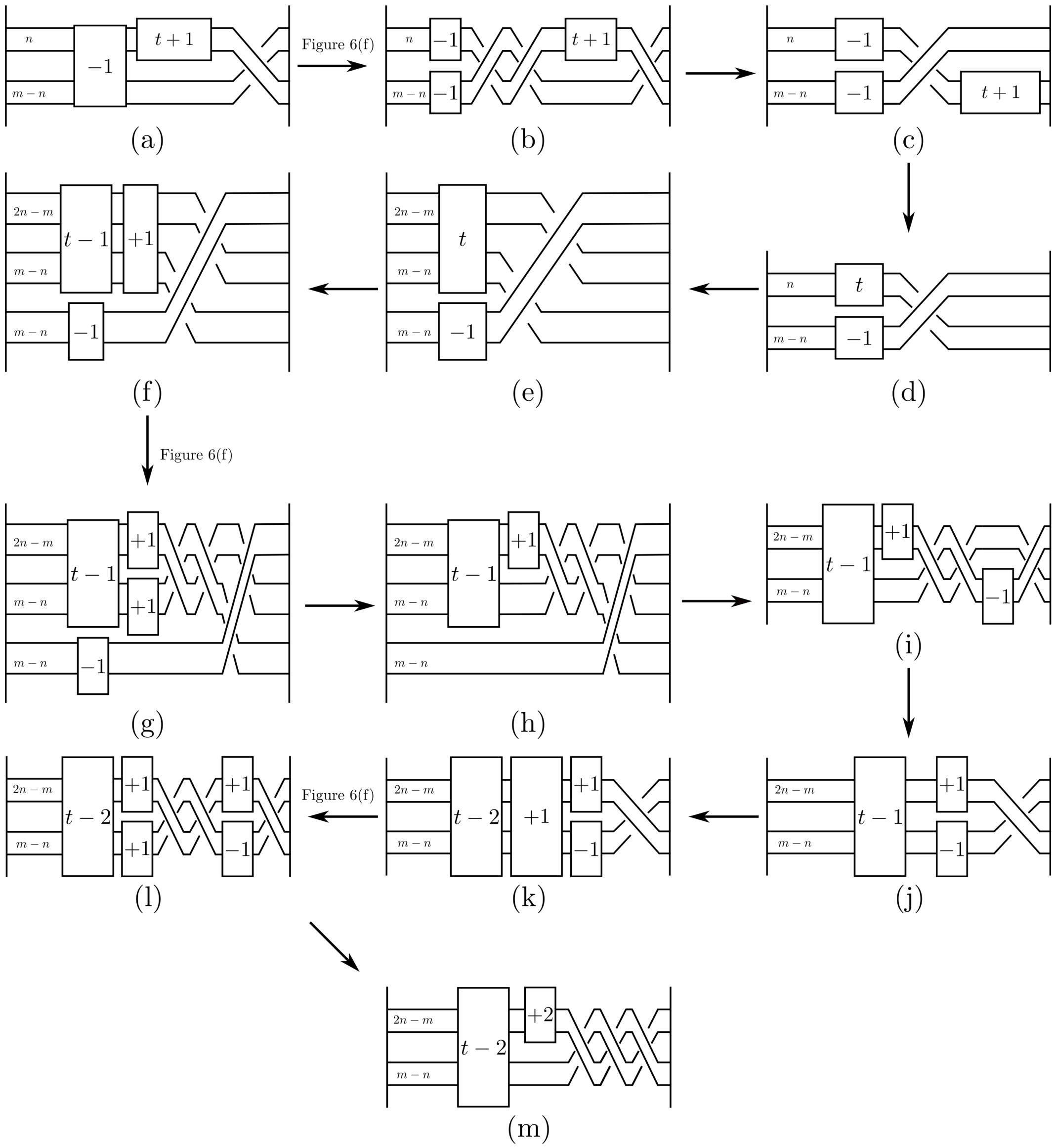

FIGURE 29.

closure of a positive or negative braid. This will be further explored in the next section but for now we observe, through Figure 31(a)-(c) that when $t = 2$ the curve is the closure of a negative braid: The braid in (a) in the figure is the braid from Figure 24(d). After applying the move in Figure 6f, and simple isotopies we obtain the braid in (c) which is clearly a negative braid when $t = 2$. $\square$

Proof of Theorem 1.3. The proof of part (1) follows from Case 1 and 2 above. Part (2)a/b follows from Case 3a/b and Case 4a above. As for part (3), observe that when $n > m$ by using Case 1 and 2 we obtain that all homologically essential curves are the closures of positive braids. When $m > n$, we have either $m - 2n > 0$ or $m - 2n < 0$. In the former case we use Case 3a and 4a to

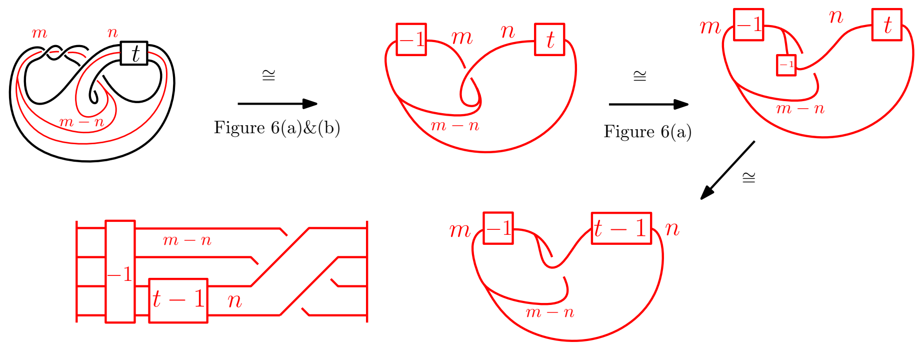

FIGURE 30.

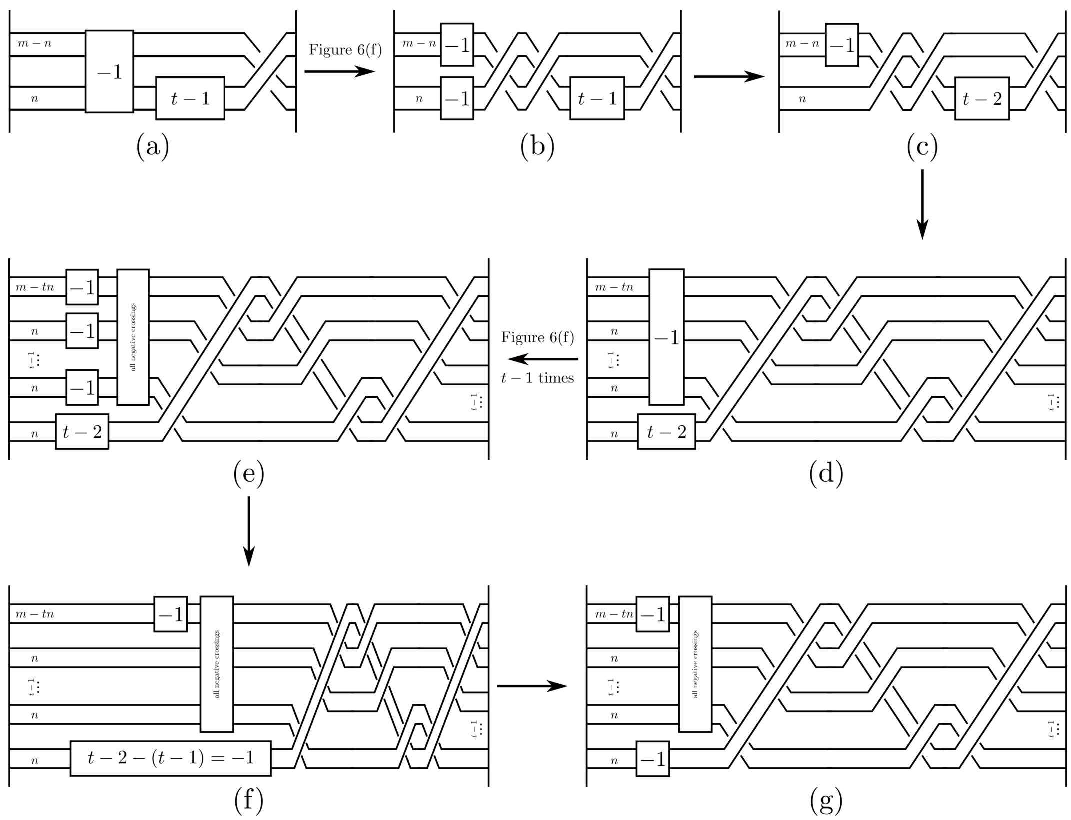

FIGURE 31.

obtain that all homologically essential curves are the closures of negative braids. In the latter case, first note that $m-2n<0$ is equivalent to $m-n<n$, Now by Case 3b all homologically essential ∞ curves are the closures of positive braids, and by Case 4b all homologically essential loop curves are the closures of negative braids. Now by using Cromwell's result and some straightforward genus calculations we deduce that when $m>n>1$ or $n>m\geq 1$ there are no unknotted curves among (positive/negative) braid closures obtained in Case $1-4$ above. Therefore, there are exactly 5 unknotted curves among homologically essential curves on Σ_K for $K = K_t$ in Theorem 1.3. □

3.4. **Twist knot with $t > 1$–Part 2.** In this section we consider twist knot $K = K_t$, $t \geq 3$, and give the proof of Theorem 1.4.

Proof of Theorem 1.4. We show that the loop curve $(3, 2)$ when $t \geq 3$ is the pretzel knot $P(2t - 5, -3, 2)$. This is explained in Figure 32. The braid in (a) is from Figure 24(d) with $m = 3, n = 2$, where we moved $(t - 2)$ full right handed twists to the top right end. We take the closure of the braid and cancel the left handed half twist on the top left with one of the right handed half twists on the top right to reach the knot in (c). In $(c) - (g)$ we implement simple isotopies, and finally reach, in (h), the pretzel knot $P(2t - 5, -3, 2)$. This knot has genus $t - 1$ ([10][Corollary 2.7]) , and so is never unknotted as long as $t > 1$. This pretzel knot is slice exactly when $2t - 5 + (-3) = 0$. That is when $t = 4$. The pretzel knot $P(3, -3, 3)$ is also known as 8_{20}. An interesting observation is that although $P(2t - 5, -3, 2)$ for $t > 2$ is not a positive braid closure, it is a quasi-positive braid closure.

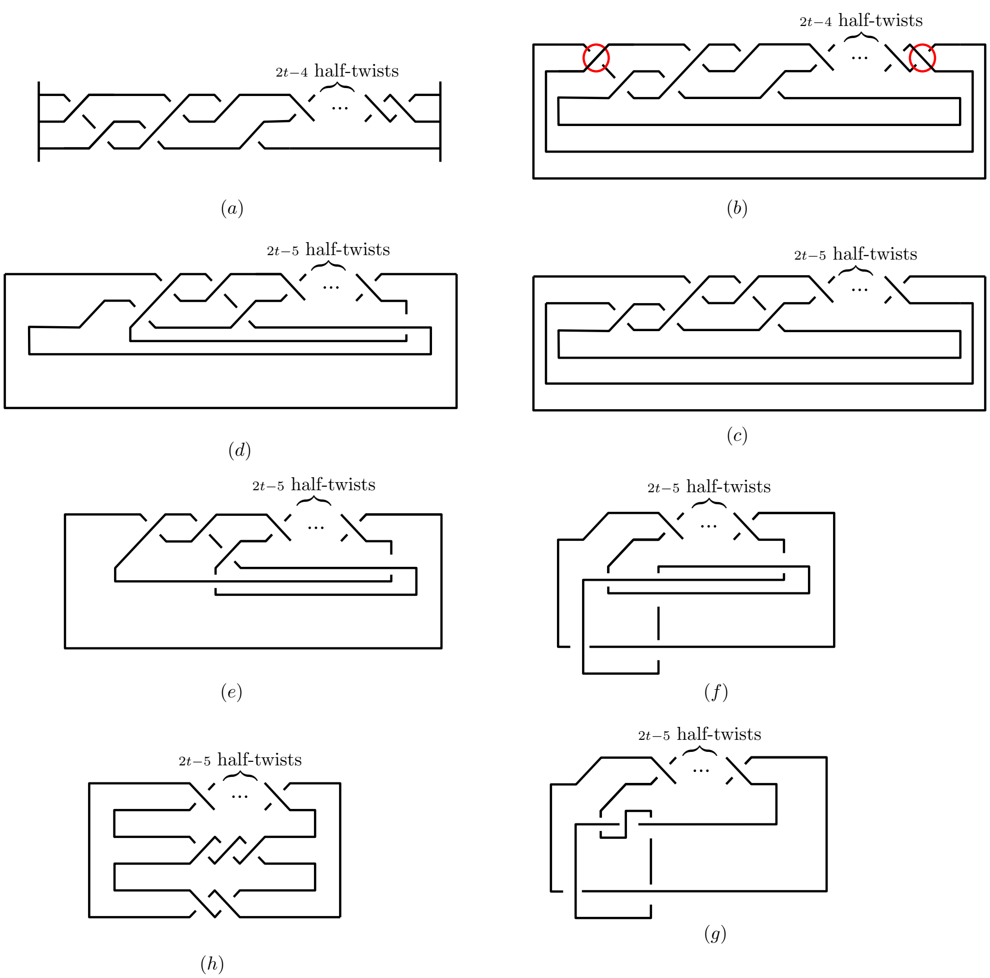

FIGURE 32.

□

Proposition 3.8. *The* (m, n) *loop curve with* $m - n = 1$, $n > 3$ *and* $t > 4$ *is never slice.*

Proof. By Rudoplh in [14], we have that for a braid closure $\hat{\beta}$ when $k_+ \neq k_-$

$$g_4(\hat{\beta}) \geq \frac{|k_+ - k_-| - n + 1}{2}$$

where β is a braid in n strands, and $k_\pm$ is the number of positive and negative crossings in β. For quasi-positive knots, equality holds. In which case, the Seifert genus is also the same as the four ball (slice) genus.

Now for the loop curve $c = (m, n)$ as in Figure 31(c), we have that

$$k_+ = (t-2)n(n-1), k_- = (m-n)(m-n-1) + 3(m-n)n$$

Hence, when $m - n = 1$, we get that $k_- = 3n$. Notice also that for $n \geq 3, t \geq 4$, we have $k_+ > k_-$. Thus, for $n > 3, t > 4, m - n = 1$ we obtain $c = \hat{\beta}$ is never slice as;

$$g_4(\hat{\beta} = c) \geq \frac{(t-2)n(n-1) - 3n - m + 1}{2} = n((t-2)(n-1) - 4) > 0$$

It can be manually checked that the $(4, 3)$ loop curve when $t = 3$ is not slice either. □

Remark 3.9. The inequality in the proof above can also be thought as a generalization to the Seifert genus calculation formula we used for positive/negative braid closures, since for those braids when, $|k_+ - k_-|$ is the number of crossings and n, the braid number, is exactly the number of Seifert circles. Thus Rudolph's inequality can also be used in the previous cases to show that there are no slice knots in the cases where we found that there are no unknotted curves

4. Whitehead Doubles

In this section we provide the proof of Theorem 1.6

Proof of Theorem 1.6. Let $f : S^1 \times D^2 \to S^3$ denote a smooth embedding such that $f(S^1 \times \{0\}) = K$. Set $T = f(S^1 \times D^2)$. Up to isotopy, the collection of essential, simple closed, oriented curves in ∂T is parameterized by

$$\{m\mu + n\lambda \mid m, n \in \mathbb{Z} \text{ and } \gcd(m, n) = 1\}$$

where μ denotes a meridian in ∂T and λ denotes a standard longitude in ∂T coming from a Seifert surface [17, 18]. With this parameterization, the only curves that are null-homologous in T are $\pm\mu$ and the only curves that are null-homologous in $S^3 \setminus \text{int}(T)$ are $\pm\lambda$. Of course $\pm\mu$ will bound embedded disks in T, but $\pm\lambda$ will not bound embedded disks in $S^3 \setminus \text{int}(T)$ as K is a non-trivial knot. In other words, the only compressing curves for ∂T in S^3 are meridians.

Suppose now that C is a smooth, simple closed curve in the interior of T, and there is a smoothly embedded 2-disk, say Δ, in S^3 such that $\partial\Delta = C$. Since C lies in the interior of T, we may assume that Δ meets ∂T transversely in a finite number of circles. Initially observe that if $\Delta \cap \partial T = \emptyset$, then we can use Δ to isotope C in the interior of T so that the result of this isotopy is a curve in the interior of T that misses a meridinal disk for T. Now suppose that $\Delta \cap \partial T \neq \emptyset$. We show, in this case too, C can be isotoped to a curve that misses a meridinal disk for T. To this end, let σ denote a simple closed curve in $\Delta \cap \partial T$ such that σ is innermost in Δ. That is σ bounds a sub-disk, Δ' say, in Δ and the interior of Δ' misses ∂T. There are two cases, depending on whether or not that σ is essential in ∂T. If σ is essential in ∂T, then, as has already been noted, σ must be a meridian. As such, Δ' will be a meridinal disk in T and C misses Δ'. If σ is not essential in ∂T, then σ bounds an embedded 2-disk, say D, in ∂T. It is possible that Δ meets the interior of D, but we can still cut and paste Δ along a sub-disk of D to reduce the number of components in $\Delta \cap \partial T$. Repeating this process yields that if C is smoothly embedded curve in the interior of T and C is unknotted in S^3, then C can be isotoped in the interior of T so as to miss a meridinal disk for T. (see [17, Theorem 9] and [12, Page 13] for a use of similar ideas).

With all this in place, we return to discuss Whitehead double of K. Suppose that F is a standard, genus one Seifert surface for a double of K. See Figure 5. The surface F can be viewed as an annulus A with a a 1-handle attached to it. Here K is a core circle for A, and the 1-handle is attached to A as depicted in Figure 33

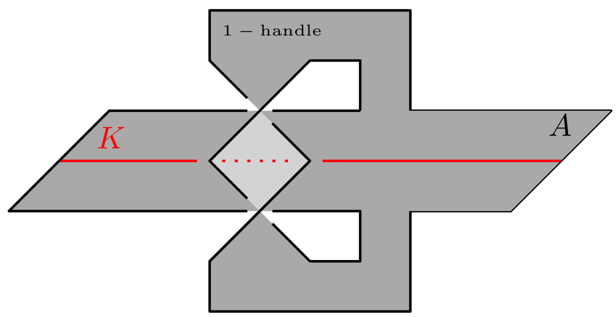

FIGURE 33. Standard genus 1 Seifert surface F for a double of K.

Observe that F can be constructed so that it lives in the interior of T. Now, the curve C that passes once over the 1-handle and zero times around A obviously misses a meridinal disk for T, and it obviously is unknotted in S^3. On the other hand, if C is any other essential simple closed curve in the interior of F, then C must go around A some positive number of times. It is not difficult, upon orienting, C can be isotoped so that the strands of C going around A are coherently oriented. As such, C is homologous to some non-zero multiple of K in T. This, in turn, implies that C cannot be isotoped in T so as to miss some meridinal disk for T. It follows that C cannot be an unknot in S^3.

□

5. CONTRACTIBLE 4-MANIFOLDS AND FINAL REMARKS

Proof of Corollary 1.9 and 1.10. In light of Theorem 1.7, the natural task is to determine self-linking number s, with respect to the framing induced by the Seifert surface, for the unknotted curves found in Theorem 1.2 and 1.6. For this we use the Seifert matrix given by $S = \left(\begin{smallmatrix} -1 & -1 \\ 0 & t \end{smallmatrix}\right)$ where we use two obvious cycles–both oriented counterclockwise–in Σ_K. Recall that, if $c = (m, n)$ is a loop curve then m and n strands are endowed with the same orientation and hence the same signs. On the other hand for ∞ curve they will have opposite orientation and hence the opposite signs. Therefore, given t, the self-linking number of $c = (m, n)$ loop curve is $s = -m^2 - mn + n^2 t$, and the self-linking number of (m, n) ∞ curve is $s = -m^2 + mn + n^2 t$. A quick calculation shows that the six unknotted curves in Figure 2 for $K_{-1} = T_{2,3}$ share self-linking numbers $s = -1, -3$. As we explained during the proof of Theorem 1.2 the infinitely many unknotted curves for the figure eight knot $K_1 = 4_1$ reduce (that are isotopic) to unknotted curves with $s = -1$ or $s = 1$. The five unknotted curves in Figure 4 for K_t, $t < -1$ or $t > 1$, share self-linking numbers $s = -1, t$ and $t - 2$ (see [3] and references therein for some relevant work). Finally, Theorem 1.4 finds a slice but not unknotted curve which is the curve $(3, 2)$ with $t = 4$. One can calculate from the formula above that this curve has self-linking number $s = 1$. Finally, the unique unknotted curve from Theorem 1.6 has self linking $s = -1$. The proofs follow as an obvious consequence of these calculations and Theorem 1.7 and its generalization in [5]. □

Next, we verify through Figure 34 that how not every essential curve on the genus one Seifert surface of a twist knot must be the closure of a positive (or negative) braid closure. For example, we will show that $(m, n) = (5, 2)$ ∞ curve on Seifert surface of the twist knot K_3 as a smooth knot is the twist knot $m(5_2)$ which is known to be not positive braid closure (e.g. via the KnotInfo database). To this end, we start with the braid as in Figure 34(a) which is the braid in Figure 27 where we substitute $m = 5, n = 2$ and $t = 3$. We then apply the move in Figure 6(f) to the full negative twist on 5 strands to obtain the braid in (b). After a cancellation between (-1) twist and $(+4)$ twist and small isotopy we get the braid in (c). We apply the move in Figure 6(f) again; This time to the full negative twist on 3 strands from the bottom to obtain the braid in (d). Small simplification gives the braid in (e). Observe that the top strands can be eliminated–here it will be easier to think the corresponding braid closure– to get the 3-braid in (f). A further simplification gives the braid in (g). We can organize and simplify this braid by canceling the half crossings encircled by red circles. This gives the braid in (h). We claim that the closure of this braid is the knot $m(5_2)$–mirror of 5_2. One can see this by taking the closure and applying simple plane isotopies. This method is quite easy (and fun) but slightly lengthier. An alternative method is to observe that this braid has braid description $-1, -2, -2, -2, -1, 2$ which we can reorder, via braid isotopy, to be $-2, -2, -2, -1, 2, -1$. Now a quick inspection in the KnotInfo database [11] shows that the knot 5_2 has braid description $1, 1, 1, 2, -1, 2$. So the closure of the braid in Figure 34 is indeed $m(5_2)$. The KnotInfo database can also be used to verify the knot $m(5_2)$ is not the closure of a positive/negative braid.

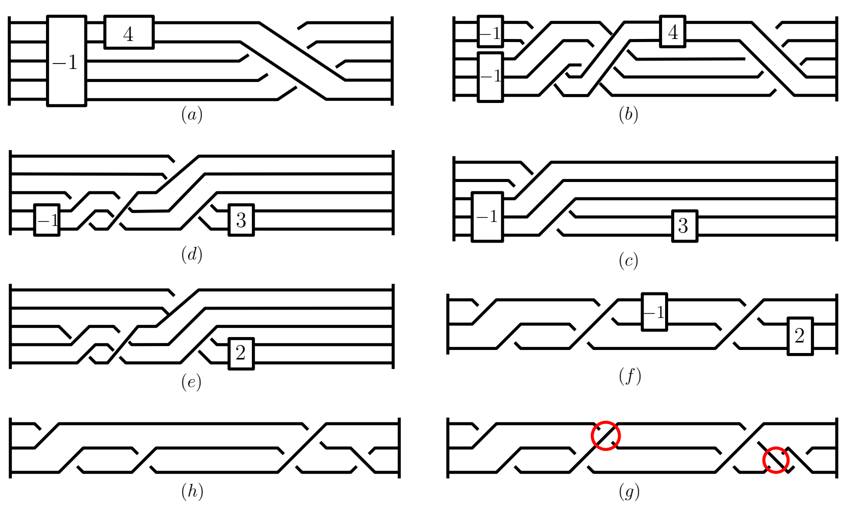

FIGURE 34. The knot $m(5_2)$ is an essential curve on genus one Seifert surface of the twist knot K_3.

REFERENCES

[1] A. Casson and J. Harer, *Some homology lens spaces which bound rational homology balls*, Pacific Journal of Mathematics **96** (1981), no. 1, 23–36.

[2] A. Casson and C. McA. Gordon, *On slice knot in dimension three*, Proc. Smpos. Pure Math. **XXXII** Amer. Math. Soc. (1978), 39–53.
[3] T. D. Cochran and C. W. Davis, *Counterexamples to Kauffman's conjectures on slice knots*, Adv. Math. **274** (2015), 263–284.
[4] P. R. Cromwell, *Homogeneous links*, J. London Math. Soc. (series 2) 39 (1989), 535–552. MR 1002465
[5] J.B. Etnyre and B. Tosun, *Homology spheres bounding acyclic smooth manifolds and symplectic fillings.* , Michigan Math. Journal **73(4)** (2023), 467–493.
[6] M. W. Hirsch, *On imbedding differentiable manifolds in euclidean space*, Ann. of Math. (2) 73 (1961), 566–571. MR 124915
[7] H. C. Fickle, *Knots, $\mathbf{Z}$-homology 3-spheres and contractible 4-manifolds*, Houston J. Math. **10** (1984), no. 4, 467–493. MR 774711
[8] R. Fintushel and R. J. Stern, *Pseudofree orbifolds*, Ann. of Math. (2) **122(2)**, (1985), 335-364–268. MR 0808222
[9] R. Kirby, *Problems in low dimensional manifold theory*, Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2, Proc. Sympos. Pure Math., XXXII, Amer. Math. Soc., Providence, R.I., (1978), pp. 273–312. MR 520548
[10] D. Kim and J. Lee, *Some invariants of pretzel links*, Bull. Austral. Math. Soc., **75** (2007), 253–271
[11] C. Livingston and A. H. Moore, *KnotInfo: Table of Knot Invariants*, knotinfo.math.indiana.edu, January 7, 2024.
[12] W. Jaco, *Lectures on three-dimensional Topology* CBMS Regional Conference Series in Mathematics, AMS **43** (1980), MR 0565450
[13] C. Manolescu, *Pin(2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture*, J. Amer. Math. Soc. **29** (2016), no. 1, 147–176. MR 3402697
[14] L. Rudoplh *Quasipositivity as an obstruction to sliceness* Bulletin of the American Mathematical Society, **29(1)**, (1993), 51–59, MR 1193540.
[15] V. A. Rohlin, *New results in the theory of four-dimensional manifolds*, Dokl. Akad. Nauk SSSR **84** (1952), 221–224. MR 0052101
[16] V. A. Rohlin, *The embedding of non-orientable three-manifolds into five-dimensional Euclidean space*, Dokl. Akad. Nauk SSSR **160** (1965), 549–551. MR 0184246
[17] D. Rolfsen, *Knots and links*, Mathematics Lecture Series, **7**, (1990) MR 1277811
[18] N. Saveliev, *Lectures on the topology of 3-manifolds*, De Gruyter Textbook, (1999), MR 1712769
[19] O. Şavk, *A survey of the homology cobordism group*, Bull. Amer. Math. Soc. (N.S.), **61(1)**, (2024), 119-157, MR 4678574.
[20] R. Stern, *Some Brieskorn spheres which bound contractible manifolds*, Notices Amer. Math. Soc (25) (1978).
[21] A. Stoimenow, *Positive knots, closed braids and the Jones polynomial*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) Vol. II, (2003) 237–285. MR 2004964
[22] B. Tosun, *Stein domains in $\mathbb{C}^2$ with prescribed boundary*, Adv. Geom. **22(1)** (2022), 9–22. MR 4371941
[23] C. T. C. Wall, *All 3-manifolds imbed in 5-space*, Bull. Amer. Math. Soc. **71** (1965), 564–567. MR 175139
[24] E. C. Zeeman, *Twisting spun knots*, Trans. Amer. Math. Soc. **115** (1965), 471–495. MR 195085

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