STEIN DOMAINS IN $\mathbb{C}^2$ WITH PRESCRIBED BOUNDARY

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ABSTRACT. This article aims to give an overview of the research related to the topological characterization of Stein domains in complex two dimensional space, and an instance of their many important connections to the smooth manifold topology in dimension four. A particular goal is going to be motivating and explaining the following remarkable conjecture.

Gompf's conjecture. No Brieskorn integral homology sphere (other than S^3) admits a pseudoconvex embedding in $\mathbb{C}^2$, with either orientation.

We also include some new examples and results that considers the conjecture for families of rational homology spheres which are Seifert fibered, and integral homology spheres which are hyperbolic.

1. INTRODUCTION

Arguably one of the most important driving forces for defining new geometric or complex analytic structures on smooth manifolds, such as Stein/Symplectic/Contact structures or integrable foliations, is the question: How much of the smooth topology of underlying manifolds can be determined by those geometric structures defined on them? Recent advances in [1, 14, 16, 17, 23, 32] provide an abundance of evidence on many fronts that this is actually already coming to fruition, and sometimes amazingly to a complete understanding [17]. This philosophy is of special importance when the underlying manifold is 4–dimensional, as despite many advances on many fronts, much of the mystery about 4–manifolds both smooth and topological is yet to be understood. For example, one giant question is about the resolution of the smooth 4–dimensional Poincaré conjecture

Conjecture 1 (4D Smooth Poincaré Conjecture)**.** *A smooth four dimensional manifold homeomorphic to the* 4*-sphere* S^4 *is actually diffeomorphic to it.*

Stein manifolds and their theory have been crucial tool to better understand smooth manifold topology [3,5,10,14,26,28]. Recall that a Stein manifold is a complex manifold with a particularly nice convexity properties, and the existence of a Stein structure on a smooth manifold is a homotopical question, though it is more subtle when underlying smooth manfold is 4–dimensional (see Section 2). So it might be supposed that with no topology to get in the way one can always find a Stein structure. In this direction we have the following persuading reformulation [57, Remark 4.8].

Theorem 2 (Eliashberg's reformulation)**.** *The smooth* 4*–dimensional Poincaré conjecture is true if and only if every contractible* 4*–manifold* X^4 *with* $\partial X^4 = S^3$ *admits a Stein structure.*

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At the moment, it is hard to justify how much this reformulation makes the original conjecture easier, but it certainly provides a new avenue to think about special cases or generalizations of the reformulation. For example, an important question that was raised in contact- and symplectic-geometric circles is the following.

Question 3. *Does every compact contractible 4-manifold with Stein fillable boundary admit a Stein structure?*

As a sample result, we mention the following result that shows the answer to Question 3 in general is negative.

Theorem 4 (Mark-Tosun [38])**.** *There exists a compact contractible manifold with no Stein structures, with either orientation.*

Our example does not have 3-sphere boundary, but illustrates that there is no general existence principle that would resolve Conjecture 1 via the reformulation mentioned above. On the other hand it is still an important question to better understand, and if possible characterize which contractible 4–manifolds admit a Stein structure. In particular, those that smoothly embedded in $\mathbb{R}^4$ as these are related to Gompf's conjecture (see Section 2 below for more details), and also to recent work of Weimin Chen [9] regarding certain questions in smooth manifold topology. In particular, Chen's work has the motivation to find smooth 4-manifolds homeomorphic but not diffeomorphic to the complex projective plane $\mathbb{C}P^2$. One of the key step in his program requires to find Stein domains in $\mathbb{C}^2$ with varying degree of convexity (see [10, 11, 39, 41]) and understand their contact boundary. Motivated by this, we will approach this question from the three-manifold topology point of view. In particular, we start with the investigation of the following broad question that sits at the intersection of Stein/Symplectic topology, low dimensional smooth topology and complex analytic theory in severable variables.

Question 5. *Which integral homology spheres embed in $\mathbb{C}^2$ as the boundary of a Stein domain in $\mathbb{C}^2$?*

We will expand on this question and a conjectural picture for its partial resolution in later sections. For now we point out that Question 5 is actually a natural extension of two classic, and well-studied problems, with tremendous impact, in low dimensional topology: Which integral homology spheres admit topological/smooth embeddings in $\mathbb{R}^4$? For the topological ones, there is a complete answer due to Freedman from 1982, who proves, for example, that any integral homology sphere bounds a contractible topological 4-manifold. The answer is negative for the smooth embeddings by many people, starting with as a consequence of the celebrated Rokhlin's Theorem from 1952. This result shows for example that the Poincaré homology sphere cannot smoothly be embedded in $\mathbb{R}^4$. This contrast of topological vs smooth embeddings yielded an unknown phenomenon: there exist closed oriented 4-manifolds with no smooth structures. On the other hand there are many infinite families of integral homology spheres (including Brieskorn homology spheres and 3-manifolds modeled on other geometries) that do embed in $\mathbb{R}^4$ smoothly. For example, smooth $\frac{1}{n}$ surgery on a slice knot, where the slice disc has two minimums, gives such homology spheres. It is worth mentioning though that despite many advances and lots of work done in the last seven decades, it is still unclear, for example, which Brieskorn homology spheres embed in $\mathbb{R}^4$ smoothly, and which do not. This line of research brought a huge increase in our understanding of smooth manifold topology. We think that Question 5, as we mentioned above and will discuss below in detail, is not just a natural next question but also has strong potential for better understanding smooth manifold topology.

Plan. We start, in Section 2, by reviewing some basics of Stein manifolds and Eliashberg's topological characterization of such manifolds. In Section 3, we explain Gompf's work towards understanding Question 5, and his remarkable prediction for this question, as Conjecture 11, as well as our partial progress, joint with T. Mark, towards answering this conjecture. In Section 4, we study some contractible 4–manifolds with hyperbolic boundary, as well as infinite families of rational homology balls with Seifert fibered boundary, and provide some new results (Theorem 18 and Theorem 20).

2. Stein manifolds and their topology

2.1. **Stein manifolds and their topology.** Recall that a *Stein manifold* is a complex manifold (W, J) admitting a proper Morse function ϕ that is bounded below and strictly plurisubharmonic (or J-convex). The J-convex function ϕ produces a Kähler form $\omega_\phi := -dd^{\mathbb{C}}\phi = -d(d\phi \circ J)$ on W, and $\phi^{-1}(-\infty, c]$ will be a Stein domain (compact parts of Stein manifolds) for any regular value c. This in turn implies that the level sets are contact manifolds where contact structure is gotten from the complex tangencies of the boundary.

Example 6. *$(\mathbb{C}^2, |z|^2, i)$ is a Stein manifold. With the given plurisubharmonic function, we get $(B^4, |z|^2, i)$ as a Stein domain. The contact boundary is (S^3, ξ_{std}) where by considering S^3 as the unit sphere in $\mathbb{R}^4$ and taking coordinates (x_1, y_1, x_2, y_2) we can describe the contact structure ξ_{std} explicitly as the kernel of $\alpha_{std} = (d|z|^2 \circ i)|_{S^3} = (x_1dy_1 - y_1dx_1 + x_2dy_2 - y_2dx_2)|_{S^3}$.*

A fundamental question one can ask about Stein manifolds/domains is how common they are and what topology they permit. Making this question more precise we can first consider Stein manifolds as abstract manifolds, and ask which open, oriented smooth $2n$–manifolds admit Stein structures? There is, in principle, a complete answer to this question. Namely, smooth manifolds supporting a Stein structure were characterized in terms of their handle structure by Eliashberg [15], with some refinements in the case of Stein surfaces due to Gompf [26].

Theorem 7 (Eliashberg)**.** *Let X be a smooth 2n-dimensional manifold with an almost complex structure J. For $n \geq 3$, if X is the interior of a possibly infinite handlebody whose indices are all less than or equal to n, then X admits a Stein structure J_0 homotopic to J. For $n = 2$, additionally, 2-handles are attached along Legendrian knots with framing $tb - 1$.*

A non-example of a Stein manifold is $S^2 \times \mathbb{R}^2$. This manifold cannot carry any finite or infinite Stein structure. This can be proved, for example, by using gauge theory. More precisely, there is a version of the adjunction inequality for Stein surfaces due to Lisca-Matić [34], which in particular implies that any homologically essential sphere in a Stein surface must have self-intersection less than or equal to -2. But $S^2 \times 0 \subset S^2 \times \mathbb{R}^2$ clearly contradicts this.

A more interesting example of a Stein domain is the following.

Example 8. *Consider the Mazur manifold as in Figure 1. One can easily see that*

- *X embeds smoothly in $\mathbb{R}^4$. To see this, we form $X \times [0, 1]$. This has the same handle structure except that the 2-handle is now attached along a knot in $S^1 \times S^3$, which we can unknot. So 1- and 2-handles geometrically cancel, and we get $X \times [0, 1] \cong B^5$. Hence, $X = X \times \{0\} \to \partial B^5 = S^4 = \mathbb{R}^4 \cup \{\infty\}$.*
- *X is abstractly Stein. For this we apply Eliashberg's Theorem 7 above. See Figure 2.*

On the other hand, we ask whether X is Stein in $\mathbb{C}^2$.

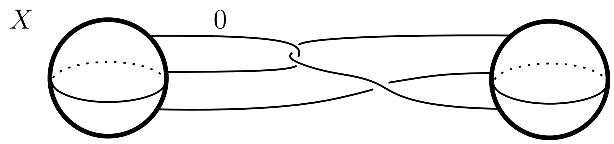

FIGURE 1. The manifold X has single $0-, 1-$ and 2-handles where the 2–handle is attached along a knot that links the 1–handle algebraically once. This is a recipe for a contractible manifold known as Mazur-type manifold.

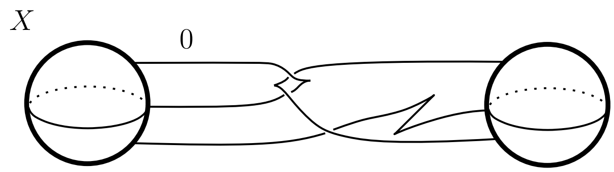

FIGURE 2. The manifold X, described as a Stein handlebody following Eliashberg's recipe in Theorem 7. The attaching sphere for the 2–handle is in Legendrian form, and one can easily calculate that its Thurston-Bennequin number is $tb = 1$.

More generally, we ask when open subsets of a fixed complex surface (e.g. $\mathbb{C}^2$) can be made Stein in the complex structure inherited from the complex surface? We will be particularly interested in compact pieces (Stein sub-domain) of such a Stein manifold. Before we further proceed we pause to provide a source for many manifolds that are abstractly Stein but not Stein as a subset of a $\mathbb{C}^2$: Let X be a 4-manifold obtained by attaching a 2–handle to B^4 along a Legendrian knot K with framing $tb - 1$, then by Eliashberg's result Theorem 7 such an X can be made Stein abstractly. On the other hand by using simple smooth topology and the adjunction inequality for Stein surfaces, one can conclude that such an X can never be made Stein in $\mathbb{C}^2$ in its usual complex structure. A precise argument for this, which is due to Gompf, goes as follows. Suppose X is Stein, then we can consider $X = 0 - h \cup 2 - h$ sitting in S^4 as $S^4 = \mathbb{C}^2 \cup \{\infty\}$. Now removing the interior of the 0-handle leaves behind the 2-handle sitting in the 4-ball. In particular, the attaching curve of the 2-handle K is slice as it bounds the core of the 2-handle. Now a slice disk in the 0-handle together with the core of the 2-handle produces a homologically non-trivial sphere of self intersection 0 in the Stein surface X. But this is in contradiction with gauge theory fact for Stein surfaces mentioned above [34]. In other words, for a given 4–manifold X in a fixed complex surface (Z, J), being abstractly Stein, though is necessary, is not sufficient to conclude that X is Stein with respect to $J_{|X}$.

3. HOLOMORPHIC CONVEXITY, GOMPF CONJECTURE AND RESULTS

We assume from now on that the ambient complex manifold is $(\mathbb{C}^n, i)$. The particular type of convexity for a Stein sub-domain X of $\mathbb{C}^n$ that we would like to consider is *holomorphic convexity*. Avoiding highly technical aspects, we say X is holomorphically convex if one can find a strictly i-convex function ψ in a neighborhood U of X. Such domains are also called domains of holomorphy in complex analytic literature. Stein domains in Question 5 are to be understood as holomorphically convex or domains of holomorphy. A recent research program of Gompf [26,27,28], which was a huge inspiration for our work here, uncovers that holomorphically convex Stein submanifolds in a complex surface are ubiquitous, and provides the following flexibility theorem (an alternate, but less explicit proof that works in any dimension can be found in Cieliebak-Eliashberg's book [10]).

Theorem 9 (Gompf)**.** *A codimension-0 submanifold U of a complex surface is isotopic to a Stein subsurface if and only if the induced complex structure is homotopic through almost-complex structures to a Stein structure on U.*

Now if W is a Mazur-type manifold and carries a Stein structure J_W. Then, as in Example 8, we can find a smooth embedding $\psi : W \longrightarrow \mathbb{C}^2$. Since W is contractible, it has only one almost-complex structure up to homotopy. So, Theorem 9 applies to isotope the embedding in $\mathbb{C}^2$ until its image is Stein. In other words we proved:

Corollary 10. *Any Mazur-type manifold which is abstractly Stein is holomorhically convex in $\mathbb{C}^2$. In particular, X in Example 8 above is holomorphically convex Stein domain in $\mathbb{C}^2$.*

Following Gompf, we refer to a 3-manifold that bounds a domain of holomorphy in $\mathbb{C}^2$ as a *pseudoconvex embedding* in $\mathbb{C}^2$. Using this result, Gompf exhibits a variety of interesting examples of Stein manifolds in $\mathbb{C}^2$ including domains of holomorphy that are diffeomorphic to non-standard smooth structures on $\mathbb{R}^4$, contractible Stein domains with non-simply-connected boundary, and domains of holomorphy that have homotopy type of S^2, disproving a conjecture of Forstnerič [22]. We note that in Gompf's flexibility theorem, there is very little control on the boundary of domain of holomorphy, and examples he has can always be shown to have hyperbolic boundary. On the other hand, Gompf in [28] makes the following remarkable prediction for Question 5.

Conjecture 11 (Gompf)**.** *No Brieskorn integer homology sphere (other than S^3) admits a pseudoconvex embedding in $\mathbb{C}^2$, with either orientation.*

Before proceeding further with what we know about this conjecture, we briefly pause to introduce Brieskorn spheres, and make some remarks for why this conjecture is not trivial for these manifolds.

3.1. **Brieskorn spheres.** The Brieskorn sphere $\Sigma(p, q, r)$ is the link of a complex hypersurface singularity. More precisely, consider the complex variety $V(p, q, r) = \{z_1^p + z_2^q + z_3^r = 0\} \subset \mathbb{C}^3$ where p, q, r are positive integers. This variety has an isolated singularity at the origin, and the intersection of $V(p, q, r)$ with a sufficiently small sphere is the 3–dimensional manifold $\Sigma(p, q, r)$. See Figure 3.

If p, q, r are pairwise relatively prime, which will be our focus in this section, then $\Sigma(p, q, r)$ is an integral homology sphere, often called Brieskorn homology sphere. Indeed, when at least one of p, q, r is 1, then $\Sigma(p, q, r) = S^3$.

By definition, the Brieskorn homology sphere $\Sigma(p, q, r)$ carries an action of the circle, which results a description as a Seifert fibered manifold, and surgery diagram, which we describe next– See [47] for more details, and the definitions and conventions we are using here.

For any pairwise co-prime integers $p, q, r \geq 2$, the Brieskorn homology sphere $\Sigma(p, q, r)$ is the unique (small) Seifert fibered homology sphere over S^2 with multiplicities p, q, r. The (unnormalized) Seifert invariants $\frac{p}{p'}, \frac{q}{q'}, \frac{r}{r'} \in \mathbb{Q}$ that determine $\Sigma(p, q, r)$, up to orientation and fiber

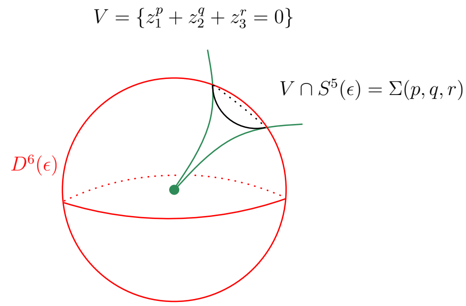

FIGURE 3. Brieskorn spheres $\Sigma(p, q, r)$. Here $S^5(\epsilon)$ denotes a sufficiently small sphere around the origin.

preserving diffeomorphism, and can be chosen arbitrary as long as they satisfy the following equation.

$$d = p'qr + pq'r + pqr' = \pm 1 \tag{1}$$

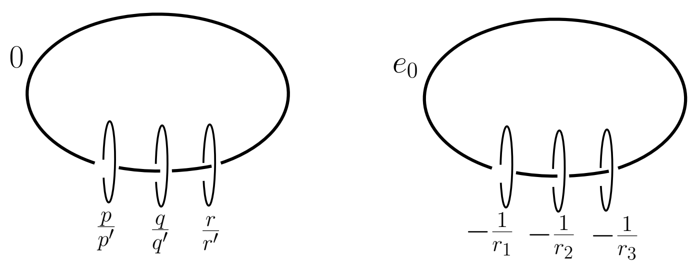

FIGURE 4. The manifold $\Sigma(p, q, r)$ with the unnormalized/normalized Seifert invariants on the left/right. We will use notation $Y(e_0; r_1, r_2, r_3)$ for the description on the right where e_0 is either -1 or -2 and $r_i \in \mathbb{Q} \cap (0, 1)$.

We can reduce Equation 1 modulo p (similarly modulo q and modulo r) to uniquely determine (p', q', r'). So, in particular we can take that $0 < |p'| < p$, $0 < |q'| < q$ and $0 < |r'| < r$ (See [47, Section 1.1.4]). A surgery description of $\Sigma(p, q, r)$ is depicted on the left hand side in Figure 4. The case $d = 1$ in Equation 1 corresponds to the usual orientation when the $\Sigma(p, q, r)$ is oriented as the link of singularity of $\{z_1^p + z_2^q + z_3^r = 0\}$ in $\mathbb{C}^3$, and $d = -1$ corresponds to the opposite orientation. In the below we fix an orientation so that $d = 1$, and let $M = \Sigma(p, q, r)$.

Using our notation above, denote the number $-(\frac{p'}{p} + \frac{q'}{q} + \frac{r'}{r})$, which is an invarinat of M, by $e(M)$, called the (rational) *Euler number* of M [43]. Using Equation 1, we get that $e(M) = -\frac{1}{pqr}$. Alternatively, by applying negative Rolfsen twists along small meridional curves in the diagram on the left hand side of Figure 4, we can replace $\frac{p}{p'}$, $\frac{q}{q'}$, $\frac{r}{r'}$ with $-\frac{1}{r_1} = \frac{p}{p''}$, $-\frac{1}{r_2} = \frac{q}{q''}$, $\frac{1}{r_3} = \frac{r}{r''}$, respectively so that $-\frac{1}{r_i} < -1$ (i.e. $r_i \in (0, 1)$) for $i = 1, 2, 3$. The resulting framing on the large unknot is $e_0(M)$, which is an invariant of M. In this case the data (e_0, r_1, r_2, r_3) correspond to the *normalized* Seifert invariants of M. We note two things: first, each negative Rolfsen twist reduces the framing 0 of the large unknot on the left hand side of Figure 4 by one. So, $e_0(M)$ is the total number of Rolfsen twists applied times -1. Second, say $\frac{p}{p'}$ is positive, then to normalize it we apply a negative Rolfsen twist, and get $\frac{p}{p'-p} = \frac{p}{p''}$. From this we see that $\frac{p''}{p} = -1 + \frac{p'}{p}$. Combining this with the definition of $e(M)$ above and Equation (1) we find the following for the normalized invariants:

$$r_1 + r_2 + r_3 = -e_0(M) - \frac{1}{pqr} = -e_0(M) + e(M).$$

So, $e_0(M) \in \{-1, -2\}$ as $0 < r_1 + r_2 + r_3 < 3$. In terms of unnormalized Seifert invariants we have $e_0(\Sigma(p,q,r))$=$\left\lfloor -\frac{p'}{p} \right\rfloor + \left\lfloor -\frac{q'}{q} \right\rfloor + \left\lfloor -\frac{r'}{r} \right\rfloor$.

For oppositely oriented manifold we can easily calculate that

$$e(-M) = -e(M) \text{ and } e_0(-M) = -3 - e_0(M).$$

Remark 12. We make the following remarks for why Conjecture 11 is non-trivial for Brieskorn spheres.

(1) Since $\Sigma(p,q,r)$ is the link of complex hypersurface singularity, it always bounds some Stein manifolds, the so called Milnor fiber. Conjecture 11 is a prediction about how small can or cannot be the topology of Stein filling of Brieskorn spheres. Singularity theory aspect of Milnor fiber is quite relevant here (see Section 3.2 for more details) but we would like to point out that on a fixed $\Sigma(p,q,r)$ there might be many Stein fillable structures that are not isotopic to Milnor fillable structure. It is this aspect that makes Conjecture 11 particularly complicated to approach.

(2) Many $\Sigma(p,q,r)$ embed smoothly in $\mathbb{R}^4$. For example, all of the Casson-Harer [8], Stern and Fickle [19] families admit such embedding.

(3) A homology sphere that embeds in $\mathbb{R}^4$ necessarily bounds an acyclic 4-manifold. Some $\Sigma(p,q,r)$ have this property, some do not.

(4) Finally Conjecture 11 is the only non-trivial case of Seifert homology spheres based on the following conjecture first by Finstushel-Stern [20] in 1985 and again by Kollár [33] in 2008.

Conjecture 13 (Fintushel-Stern and Kollár). *A Seifert homology sphere with more than* 3 *singular fibers cannot bound an acyclic 4-manifold.*

As a start, we consider Conjecture 11 in the context of the family of Brieskorn spheres $\Sigma(2,3,6m\pm 1)$. For many members of this ubiquitous family, the conjecture can be answered affirmatively by purely topological means. For example, we have:

- For odd m, both $\Sigma(2,3,6m-1)$ and $\Sigma(2,3,6m+1)$ have nontrivial Rokhlin invariant, hence neither bounds an acyclic 4-manifold. In particular these manifolds do not admit even a smooth embedding in $\mathbb{C}^2$.
- For even m, the Brieskorn sphere $\Sigma(2,3,6m-1)$ has $R = 1$, where R is the invariant of Fintushel-Stern. By Theorem 1.1 of [20], it follows that none of these manifolds bound acyclic 4-manifolds either. Indeed, whenever Brieskorn sphere $\Sigma(p,q,r)$, as a Seifert fibered space, has $e_0 = -2$, then by a calculation of Neumann and Zagier in [42] it has $R = 1$. So, it never bounds an acyclic 4-manifold. Moreover, according to [30, Corollary 3], such a manifold cannot even bound a rational homology ball.

This leaves the family $\Sigma(2,3,12n+1)$, for which no standard invariants (e.g., Rokhlin or $\bar{\mu}$, Fintushel-Stern's R, the Heegaard-Floer d-invariant, or Manolescu's lift β of the Rohlin invariant [36]) obstruct an acyclic filling. In fact it is known by work of Akbulut-Kirby [2] and Casson-Harer [8] in the case $n=1$, and Fickle [19] when $n=2$, that both $\Sigma(2,3,13)$ and $\Sigma(2,3,25)$ are the boundaries of smooth contractible 4-manifolds each of which has a handle decomposition with one 1-handle and one 2-handle. In particular, both of these admit smooth embeddings in $\mathbb{C}^2$. For $n \geq 3$, neither an acyclic 4-manifold bounding $\Sigma(2,3,12n+1)$ nor an embedding in $\mathbb{C}^2$ appear to be known, but it seems plausible that both exist.

Mark and the author in [38] provided the first verification of Conjecture 11 for a family of Brieskorn spheres of which some are known to admit a smooth embedding in $\mathbb{C}^2$. More precisely we proved:

Theorem 14 (Mark-Tosun)**.** *Suppose X is a smooth compact oriented acyclic 4-manifold whose boundary is the Brieskorn sphere $-\Sigma(2,3,12n+1)$, for some $n \geq 1$. Then X cannot be made symplectic that weakly fills a contact structure on its boundary.*

Note that $\Sigma(2,3,12n+1)$, with either orientation, does admit Stein fillings. As perhaps the first evidence for Conjecture 11, we have:

Corollary 15 (Mark-Tosun)**.** *No member of the family $-\Sigma(2,3,12n+1)$ admits a pseudoconvex embedding in $\mathbb{C}^2$.*

On the other hand for correcty oriented Brieskorn homology sphere $\Sigma(2,3,12n+1)$, Mark and the author proved the following result, where the proof is carried by carefully studying certain complex surfaces and making use of an interplay between Symplectic/contact geometry, gauge theory and 3/4-manifold topology.

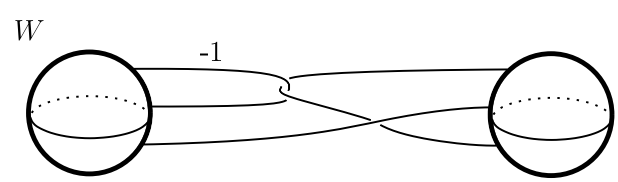

FIGURE 5. The first example of a contractible manifold with no Stein structure with either orientation.

Theorem 16 (Mark-Tosun)**.** *Let W denote the contractible 4–manifold constructed by Akbulut-Kirby in [2], see Figure 5, with boundary the Brieskorn sphere $\Sigma(2,3,13)$. Then W does not admit any Stein structures, with either orientation.*

3.2. **Gompf's conjecture: Singularity theory aspect.** There is also the rich singularity theory aspect of the research program above, which, if developed, will provide further evidence for Conjecture 11. We briefly explain this and make the connection. Recall that the link of a normal complex surface singularity—for example a Brieskorn manifold—admits a natural contact structure known as the *Milnor fillable* contact structure. Analytic smoothings of the singularity give rise to Stein fillings of this contact structure, and following Némethi and Popescu-Pampu (c.f. [40, Section 1.3]) it is natural to ask whether this construction produces *all* symplectic fillings of the Milnor fillable contact structure.

Question 17. *Is there a bijective correspondence between the Milnor fibers of the smoothing components of a normal surface singularity and the symplectic fillings of the corresponding contact boundary of the singularity?*

This correspondence was shown to hold for cyclic quotient singularities in [40], and extended to all quotient singularities by Park–Park–Shin–Urzúa in [44]. The latter makes a heavy use of algebraic geometry. The singularity corresponding to $\Sigma(p,q,r)$ is not rational, and hence does not admit an acyclic smoothing. In particular a "positive" answer to Conjecture 11 will provide strong evidence that the answer to Question 17 is YES if the Milnor fillable structure is unique (which the case for $\Sigma(2,3,12n+1)$). Conversely, a positive answer to Question 17 together with the fact that analytic smoothing cannot produce acyclic fillings of the link would immediately verify Conjecture 11 for $\Sigma(2,3,12n+1)$ (or any $\Sigma(p,q,r)$ that has a unique tight contact structure). We want to add that in a recent preprint, Plamanevskaya and Starkston [45] show that the answer to Question 17 is negative for certain rational singularities.

4. More Examples

4.1. **Non-Stein contractible 4–manifold with hyberbolic boundary.** In this section we give an example of a non-Stein contractible manifold with hyperbolic boundary. Consider the contractible manifold in Figure 6. From its handle description, one can easily calculate that the 2–handle is attached along a knot for which the obvious Legendrian representative has $tb = -5$, which makes it difficult to apply Eliashberg's result in Theorem 7. Of course, it is possible that this knot has another Legendrian representative for which $tb = 0$, but it is not obvious how to find such a representative. Our result below yields immediately that such a representative cannot exist.

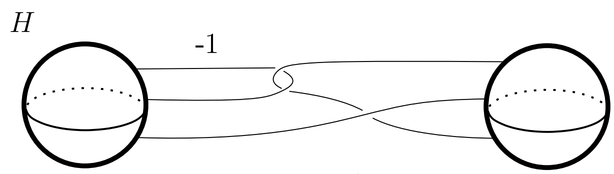

FIGURE 6. A non-Stein contractible 4–manifold. Its boundary is diffeomorphic to Y_2, the 3–manifold obtained by smooth $-\frac{1}{2}$ surgery on the figure-eight knot.

Theorem 18. *Let Y_n denote the 3–manifold obtained by smooth $-\frac{1}{n}$ surgery on the figure-eight knot. Suppose X is a smooth compact oriented acyclic 4-manifold whose boundary is Y_n, for some $n \geq 1$. Then X does not admit a symplectic structure weakly filling a contact structure on its boundary. In particular, the contractible 4–manifold H in Figure 6 does not admit any Stein structures.*

The reason for this as we will explain is entirely due to the contact topology of the boundary.

Proof. The classification of exceptional (i.e. non-hyberbolic) surgeries on the figure-eight knot is completed in [7, Theorem $1.1(4)$]. It follows from this classification that the manifold Y_n, see Figure 7, is a hyperbolic manifold for any $n \neq 0, 1, -1$ (the case $n = 1$ corresponds to $-\Sigma(2,3,7)$ and this is already covered in Theorem14). Recall that a Stein domain induces a natural tight contact structure ξ on its boundary Y as the field of complex tangencies. The homotopy class of an oriented tangent 2-plane field ξ on a homology sphere Y is determined by an invariant $\theta(\xi) =$

$c_1^2(X,J)-2\chi(X)-3\sigma(X) \in \mathbb{Z}$. Thus, if (Y,ξ) is the contact boundary of an acyclic Stein manifold (X,J), then necessarily $\theta(\xi)=-2$. The same conclusion holds if (Y,ξ) is weakly symplectically filled by an acyclic manifold (X,ω), since we can select an almost-complex structure compatible with both ω and ξ. In other words, if Y is an integral homology sphere, then the classification of tight contact structures on Y, if possible to determine, is a crucial first step to understand whether Y bounds an acyclic weakly symplectic 4-manifold or not. Now in our case, Y_n is a homology sphere, and the classification of tight contact structures on Y_n for $n>0$ is recently produced in [12]. According to this classification, Y_n admits exactly two tight structures both of which are Stein fillable. Moreover, it was further proved that these contact structures have non-vanishing *reduced* Heegaard-Floer contact classes. The reduced contact class lives in $HF^+_{red}(-Y_n)$, and this group is calculated in [53] to be $\mathbb{Z}^n_{(-1)}$ where the subscript is the degree in which the contact class is supported. Since the contact structures we are interested in have torsion Chern classes, their contact invariant lies in degree $-\frac{\theta}{4}-\frac{1}{2}$ [54]. But this degree must be -1. So we conclude that $\theta=2$ is the only possible value for tight structures on Y_n. Hence there cannot exist an acyclic weakly symplectic manifold that fills Y_n. This finishes the first part of the theorem. Finally, observe that Y_n, as a 3–manifold is diffeomorphic to a manifold obtained by smooth -1 surgery along the twist knot K_n, see Figure 7. To see this simply blow up the crossing in the figure-eight knot to convert it to a surgery on a symmetric link with framings $-\frac{1}{n}$ and -1. Swapping these framings and applying n positive Rolfsen twists gives the other description of Y_n that we are claiming. When $n=2$, K_2 is Stevedore's knot 6_1, which is the only slice twist knot. Now a simple handle calculus proves that ∂H is obtained by smooth -1 surgery along 6_1 knot which is Y_2. Hence H cannot carry any Stein (even weakly symplectic) structure. □

Remark 19. We make a few remarks about Y_n and their smooth/symplectic fillings.

(1) Observe that the manifold Y_n bounds a smooth rational homology ball for any n. To see this we describe Y_n as a surgery on a link: 0 surgery on the figure eight knot, and n surgery on its meridian. Since the figure-eight knot is rationally slice, that is it bounds a smooth disk in a rational ball filling of S^3, the latter description of Y_n gives that Y_n is the boundary of the 4–manifold obtained by attaching a 2-handle in the complement of rational slice disk. Such a 4–manifold must be a rational ball itself, which can be verified by simple homology calculations. But by the same argument as in the proof above above such a rational homology ball for $n>0$ cannot be Stein. For $n=-1$, one can use a Donaldson type argument to conclude that any rational homology ball bounded by $Y_{-1}\cong\Sigma(2,3,7)$ must have a 3–handle in it. More precisely, by [21, Lemma 2.1] Y_{-1} embeds in a $K3$ surface M, splitting it into submanifolds $M=X\cup Y$ with intersection forms $I_X\cong E_8\oplus H$ and $I_Y\cong E_8\oplus 2H$. Now if Y_{-1} were to bound a rational homology ball Z without 3-handles, turning this handle decomposition over, Z can be obtained from ∂Z by attaching handles of index ≥ 2. In particular there is a surjection $\pi_1(\partial Z)\to\pi_1(Z)$, which would imply the manifold $M'=(M\setminus X)\cup Z$ is a simply connected smooth 4-manifold with intersection form $E_8\oplus 2H$ contradicting Donaldson's result [13, Theorem B, C]. Hence, Y_{-1} cannot bound a rational homology ball carrying a Stein structure. But this argument does not generalize to other $n<0$ values, even though explicit rational balls we know in some other cases also have 3–handles in them [4].

(2) By using surgery formula for Rokhlin invariant, one can show the manifold Y_n can never bounds a smooth acyclic manifold when n is odd. On the other hand, as in Figure 6, Y_2

bounds a Mazur-type contractible manifold. At the moment, for $n > 2$ even, neither an acyclic or contractible manifold bounding Y_n is known nor any of standard homology cobordism invariants obstructs an acyclic filling.

(3) Finally, if we switch the orientation of the contractible manifold H, then we can easily equip it with a Stein structure, as Eliashberg's result Theorem 7 easily applies in that case. So, the claim in Gompf's conjecture about "either orientation" must be specific to Seifert homology spheres.

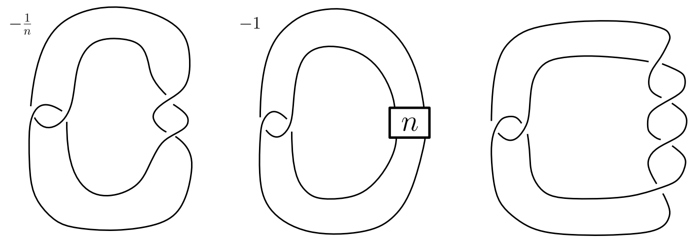

FIGURE 7. The 3–manifolds on the left and the middle are diffeomorphic. On the left is the 3-manifold obtained by smooth $-\frac{1}{n}$ surgery on the figure eight knot and the middle is the 3-manifold obtained by smooth -1 surgery along twist knot K_n, where n stands for n right handed full twists if $n > 0$ and n left handed full twists if $n < 0$. On the right is the Stevedore's knot 6_1, which is the only slice twist knot.

4.2. **Non-Stein Rational Homology Balls.** In this section we provide two infinite families of rational homology spheres that have Stein fillable boundary and bound smooth rational homology balls but cannot bound any such balls which carries a Stein structure.

Let $M_{p,p,2}$ and $M_{p,p,p}$ denote the Seifert fibered manifolds which are rational homology spheres depicted in Figure 8 (note that these 3-manifolds are the same when $p = 2$). By performing handle calculus we can easily see explicit rational homology balls, see Figure 9, that smoothly fill $M_{p,p,2}$ and $M_{p,p,p}$ – In Figure 10 we describe the explicit handle calculus showing $\partial Q_{p,p,p} \cong M_{p,p,p}$. The other case is identical. The manifolds $M_{p,p,2}$ and $M_{p,p,p}$ were also listed by Casson and Harer in [8] that they bound rational homology balls. More specifically, let $s = -1$ in [8, Theorem-(5)] for $M_{p,p,2}$ and $s = 0, q = p$ in [8, Theorem-(3)] for $M_{p,p,p}$.

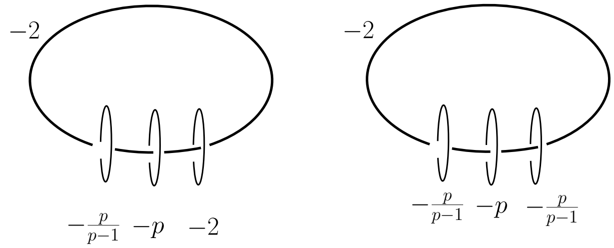

FIGURE 8. The 3–manifolds $M_{p,p,2}$ and $M_{p,p,p}$ described as Seifert fibered spaces with normalized Seifert invariants.

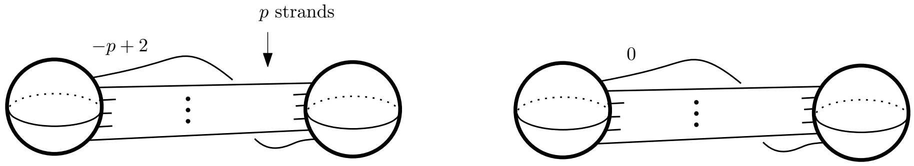

FIGURE 9. On the left is the rational homology ball $Q_{p,p,2}$ that smoothly fills $M_{p,p,2}$. On the right is the rational homology ball $Q_{p,p,p}$ that smoothly fills $M_{p,p,p}$.

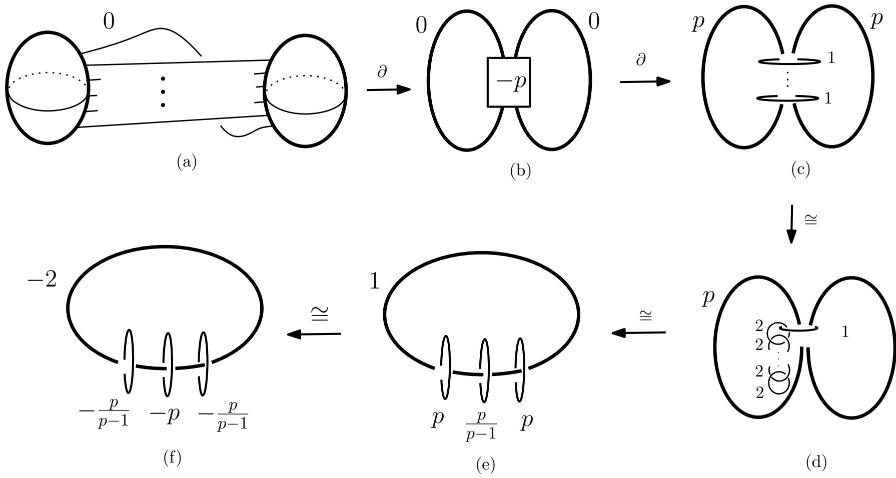

FIGURE 10. On the left is the rational homology ball $Q_{p,p,p}$. Replacing the 1-handle with 0-framed 2-handle we obtain the 2-handlebody in (b). Applying p blow ups yields (c). Applying successive handleslides yields (d). Applying the operation of slum-dunk successively yields (e). Finally applying three negative Rolfsen twists yields the drawing in (f) which is $M_{p,p,p}$.

Theorem 20. *Let X is a smooth compact oriented rational homology ball whose boundary is $M_{p,p,2}$ or $M_{p,p,p}$ for some p. Then X does not admit a symplectic structure weakly filling a contact structure on its boundary for any integer $p \geq 2$.*

The proof is a corollary of the following lemma.

Lemma 21. *For each positive integer p,*

(1) *The manifold $M_{p,p,2}$ supports exactly $p-1$ distinct isotopy classes of tight contact structures, ξ_i where $-(p-2) \leq i \leq p-2$ and $i \equiv p$ (mod 2) , which all are Stein fillable and have $\theta(\xi_i) = \frac{p^3-(p+2)i^2}{p^2}$.*

(2) *The manifolds $M_{p,p,p}$ support exactly $p-1$ distinct isotopy classes of tight contact structures, ξ_i where $-(p-2) \leq i \leq p-2$ and $i \equiv p$ (mod 2), which all are Stein fillable and have $\theta(\xi_i) = \frac{2(p^2-p-i^2)}{p}$.*

Indeed, assuming the truth of Lemma 21 for the moment, we can easily check that none of the tight contact structures ξ_i realizes the value $\theta = -2$. In particular there is no tight contact structure on $M_{p,p,p}$ or $M_{p,p,2}$ that can have $\theta = -2$ as its 3–dimensional homotopy invariant. Now if X were a rational homology ball that was a weak symplectic filling of $M_{p,p,2}$ (or on $M_{p,p,p}$), with some compatible almost complex structure J, then it would induce a tight structure ξ on $M_{p,p,2}$ (or on $M_{p,p,p}$) with

$$\theta(\xi) = c_1^2(X,J) - 2\chi(X) - 3\sigma(X) = -2$$

But as we derived using Lemma 21 that there is no such tight structure on $M_{p,p,2}$ (or on $M_{p,p,p}$), yielding the claim in Theorem 20.

Remark 22. One can easily show that the rational homology ball $Q_{p,p,p}$ depicted in Figure 9 smoothly embeds in $\mathbb{R}^4$. To see this, attach a 2-handle along a trivial curve running over the 1-handle with framing 0. Then this new 2-handle and the 1-handle of $Q_{p,p,p}$ cancel each other, leaving behind a 2-handle attached along a 0-framed unknot. Attaching a 3-handle to this yields $B^4 \subset \mathbb{R}^4$. On the other hand, the rational homology ball $Q_{p,p,2}$ cannot even smoothly embed in $\mathbb{R}^4$ when $p \neq 2$. This is because its boundary $M_{p,p,2}$ has $H_1(M_{p,p,2};\mathbb{Z}) \cong \mathbb{Z}/p^2\mathbb{Z}$ when p is odd (for example $M_{3,3,2}$ is diffeomorphic to a manifold obtained by smooth $\frac{9}{2}$ surgery on the positive trefoil knot), and $H_1(M_{p,p,2};\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z} \oplus \mathbb{Z}/(p^2/2)\mathbb{Z}$ when p is even. But a result due to Hantzsche from 1938 [31] says that if a 3–manifold Y embeds in S^4, then the torsion part of $H_1(Y)$ must be of the form $G \oplus G$ for some finite abelian group G. So, when $p \neq 2$, the manifold $M_{p,p,2}$ cannot embed in $\mathbb{R}^4$ nor can $Q_{p,p,2}$.

Proof of Lemma 21. One can easily check that $M_{p,p,p}$ and $M_{p,p,2}$ are L-space for any positive integer p [35, Theorem 1.1], and the classification of tight contact structures on such manifolds is completed by Ghiggini in [24] (see also [50]). According to that classification, $M_{p,p,2}$ and $M_{p,p,p}$ each have exactly $p-1$ tight contact structures, which are all Stein fillable. We can explicitly list and describe these Stein fillable structures as follows. Consider first the 4–manifold $X_{p,p,p}$ with $\partial X_{p,p,p} = M_{p,p,p}$, which is given as a link surgery as in Figure 11. We convert this to a Legendrian surgery on all possible Legendrian realizations of the link, where each component is Legendrian unknot. We easily see that the only component of this Legendrian link that will require stabilizations is the one with framing $-p$, and for which $p-2$ stabilizations are needed. This stabilizations can be done in $p-1$ ways, yielding a Legendrian unknot K_i with rotation number $\mathrm{r}(K_i) = i$ where $-(p-2) \leq i \leq (p-2)$ and $i \equiv p \pmod 2$. From this we obtain Stein domain $(X_{p,p,p}, J_i)$, and hence $p-1$ Stein fillable tight contact structures ξ_i on the boundary $M_{p,p,p}$. Finally, by a result of Lisca-Matić these contact structures are pairwise different, which yields the exact number of tight contact structures on $M_{p,p,p}$. Moreover from this description we can calculate the characteristic values of the Euler characteristic, signature and the square of the first Chern class of $(X_{p,p,p}, J_i)$ as follows.

First from the handle diagram in Figure 11, it is easy to calculate that $\chi(X_{p,p,p}) = 2p+1$. Next since $M_{p,p,p}$ is an L-space, by a result of Ozsváth-Szabó [55, Theorem 1.4.] any symplectic filling of $M_{p,p,p}$ is negative definite. In particular, $X_{p,p,p}$ is negative definite, and hence $\sigma(X_{p,p,p}) = -2p$. Finally, following Gompf [26], we can calculate the square of the first Chern class as

$$c_1^2(X_{p,p,p}, J_i) = \vec{\mathrm{r}}^{\,T} I_p^{-1} \vec{\mathrm{r}} \tag{2}$$

where

$$\vec{\mathrm{r}} = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & \cdots & 0 & \mathrm{r}(K_i) \end{bmatrix}^T$$

is the rotation vector and I_p is the intersection matrix given as:

$$I_p = \begin{bmatrix}
-2 & 1 & & & & & 1 & & & & 1 \\
1 & -2 & 1 & & & & & & & & \\
 & 1 & -2 & 1 & & & & & & & \\
 & & & & \ddots & & & & & & \\
 & & & & 1 & -2 & & & & & \\
1 & & & & & & -2 & 1 & & & \\
 & & & & & & 1 & -2 & 1 & & \\
 & & & & & & & & \ddots & & \\
 & & & & & & & & 1 & -2 & \\
1 & & & & & & & & & & -p
\end{bmatrix}$$

We note that, when forming the intersection matrix I_p of $X_{p,p,p}$, we started with the central curve, then ran through the chain of -2 spheres on the left and the chain of -2 spheres on the right and finally $-p$ sphere in the middle of the Figure 11. Now using the fact that the rotation vector $\vec{\mathrm{r}}$ can be non-zero only in one entry, we find it easier to solve the linear system $I_p\vec{x} = \vec{\mathrm{r}}$ for $\vec{x} = [a, x_1, \cdots, x_{p-1}, y_1, \cdots, y_{p-1}, z]^T$. In fact, we only need to know the term z in this vector for our final calculation. The liner system $I_p\vec{x} = \vec{\mathrm{r}}$ expands as

$$\begin{cases}
-2a + x_1 + y_1 + z = 0 \\
a - 2x_1 + x_2 = 0 \\
x_1 - 2x_2 + x_3 = 0 \\
\cdots \\
x_{p-2} - 2x_{p-1} = 0 \\
a - 2y_1 + y_2 = 0 \\
y_1 - 2y_2 + y_3 = 0 \\
\cdots \\
y_{p-2} - 2y_{p-1} = 0 \\
a - pz = \mathrm{r}(K)
\end{cases}$$

Now some easy calculations shows that $z = -\frac{2\,\mathrm{r}(K)}{p}$. As explained above, the Legendrian knot in Figure 11 need to be stabilized $p-2$ times, yielding Legendrian knot K_i with rotation number $\mathrm{r}(K_i) = i$ where $-(p-2) \leq i \leq (p-2)$ and $i \equiv p \pmod 2$. Substituting this in Equation 2, we get:

$$c_1^2(X_n, J_i) = \vec{\mathrm{r}}^T I_p^{-1} \vec{\mathrm{r}} = \vec{\mathrm{r}}^T \vec{x} = -\frac{2i^2}{p}.$$

Thus Gompf invariant of ξ_i, for $-(p-2) \leq i \leq (p-2)$ and $i \equiv p \pmod 2$, can be calcuted to be

$$\theta(\xi_i) = \frac{2(p^2 - p - i^2)}{p}.$$

From this, we can easily see that none of the tight contact structures realizes the value of $\theta = -2$.

In a quite similar way, we can list exactly $p-1$ tight structures for ξ_i on $M_{p,p,2}$, where $-(p-2) \leq i \leq (p-2)$ and $i \equiv p \pmod 2$, from its Stein fillings $(X_{p,p,2}, J_i)$. From this, again similar to the case above we can easily calculate the characteristic values of $X_{p,p,2}$ and use those to compute that $\theta(\xi_i) = \frac{p^3-(p+2)i^2}{p^2}$, and see immediately that none realizes $\theta = -2$ value.

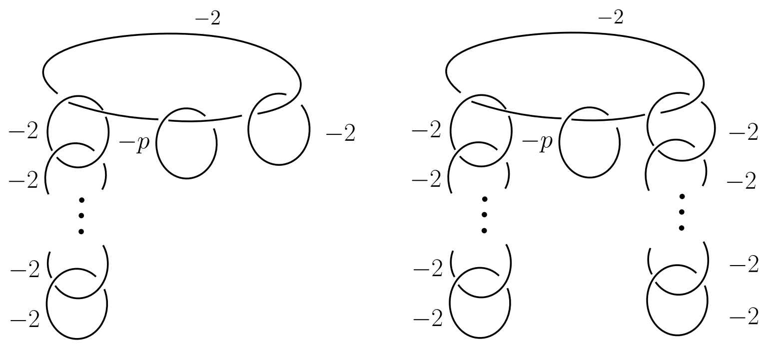

FIGURE 11. On the left is the 4–manifold $X_{p,p,2}$ with $\partial X_{p,p,2} = M_{p,p,2}$, and on the right $X_{p,p,p}$ with $\partial X_{p,p,p} = M_{p,p,p}$. In each figure the number of -2 framed unknots in each of the chain is $p-1$.

□

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REFERENCES

1. M. Abouzaid. Framed bordism and Lagrangian embeddings of exotic spheres. *Ann. Math.* **2**, 71-185, 2012.
2. S. Akbulut and R. Kirby. Mazur manifolds. *Michigan Math. J.*, **26**(**3**) 259-284, 1979.
3. S. Akbulut. A fake compact contractible 4-manifold. *J. Differential Geom.* **33**, 335-356, 1991.
4. S. Akbulut and K. Larson. Brieskorn spheres bounding rational balls. *Proc. Amer. Math. Soc.* **146**(**4**), 1817-1824, 2018.
5. S. Akbulut and R. Matveyev. A convex decomposition theorem for four-manifolds. *Internat. Math. Res. Notices* **1998**, 371-381, 1998.
6. S. Akbulut and K. Yasui. Cork twisting exotic Stein 4-manifolds. *J. Differential Geom.* **93**, 1-36, 2013.
7. M. Brittenham and Y. Q. Wu. The classification of exceptional Dehn surgeries on 2-bridge knots . *Comm. Anal. Geom.* (2) **9**(**1**), 97-113, 2001.
8. A. J. Casson and J. L. Harer. Some homology lens spaces which bound rational homology balls. *Pacific J. Math.*, **96**(**1**), 23-36, 1981.

9. W. Chen. Contact splitting of symplectic $\mathbb{Q}$-homology $\mathbb{C}P^2$ *Proceedings of the Gökova Geometry-Topology Conference 2017*, Int. Press, Somerville, MA, 53-72, 2018.
10. K. Cieliebak and Y. Eliashberg. From Stein to Weinstein and back. *Colloquium Publications* **59** Amer. Math. Soc. Providence, 2012.
11. K. Cieliebak and Y. Eliashberg. The topology of rationally and polynomially convex domains. *Invent. Math.* **199** 215-238, 2015.
12. J. Conway and H. Min. Tight contact structures on surgeries on the figure-eight knot. Preprint, ArXiv:1901.06066.
13. S.K. Donaldson. Connections, cohomology and the intersection forms of 4-manifolds. *J. Diff. Geom.* (2) **24**, 275-341, 1986.
14. Y. Eliashberg. Contact 3-manifolds twenty years since J. Martinet's work. *Geometry of low dimensional manifolds, volume II* (Ed. Donaldson and Thomas), *Cambridge*, 1990.
15. Y. Eliashberg. Topological characterization of Stein manifolds of dimension > 2. *Internat. J. Math.*, $\mathbf{1}(\mathbf{1})$, 29-46, 1990.
16. T. Ekholm and I. Smith. Exact Lagrangian immersions with a single double point. *J. Amer. Math. Soc.* $\mathbf{29}(\mathbf{1})$, 1-59, 2016
17. T. Ekholm, L.N. Ng and V. Shende. A complete knot invariant from contact homology. Preprint, ArXiv:1606.07050
18. J. B. Etnyre and K. Honda. Tight contact structures with no symplectic fillings. *Invent. Math.*, $\mathbf{148}(\mathbf{3})$, 609–626, 2002.
19. H. F. Fickle. Knots, Z-homology 3-spheres, and contractible 4-manifolds. *Houston J. Math.*,$\mathbf{10}(\mathbf{4})$, 467-493, 1984.
20. R. Fintushel and R. J. Stern. Pseudofree orbifolds. *Ann. of Math* (2) $\mathbf{122}(\mathbf{2})$, 335-364, 1985.
21. R. Fintushel and R. J. Stern. Homotopy $K3$ surfaces containing $\Sigma(2,3,7)$. *J. Diff. Geom.* (2) **34**, 255-265, 1991.
22. F. Forstnerič. Complex tangents of real surfaces in complex surfaces. *Duke Math. J.* **67**, 353–376, 1992.
23. D. Gabai. Foliations and the topology of 3-manifolds, *J. Diff. Geom.* **18** 445–503, 1983.
24. P. Ghiggini. On tight contact structures with negative maximal twisting number on small Seifert manifolds. *Algebr. Geom. Topol* $\mathbf{8}(\mathbf{1})$ 381–396, 2008.
25. E. Giroux. Structures de contact en dimension trois et bifurcations des feuilletages de surfaces. *Invent. Math.* **141**,615-689, 2000.
26. R. E. Gompf. Handlebody construction of Stein surfaces. *Ann. of Math.* **148**, 619-693, 1998.
27. R. E. Gompf. Constructing Stein surfaces after Eliashberg. *New perspectives and challenges in symplectic field theory*, CRM Proceedings and Lecture Notes **49** (American Mathematical Society, Providence, RI), 229-249, 2009.
28. R. E. Gompf. Smooth embeddings with Stein surface images. *J. Topol.* **4**, 915-944, 2013.
29. R. E. Gompf. Infinite order Corks. *Geom. Topl.* **4**, 915-944, 2017.
30. A. Issa and D. McCoy Smoothly embedding Seifert fibered spaces in S^4, ArXiv: 1810.04770, 2018.
31. W. Hantzsche. Einlagerung von Mannigfaltigkeiten in euklidische Räume, *Math. Z.* $\mathbf{43}(\mathbf{1})$, 38–58, 1938.
32. P. B. Kronheimer and T.S. Mrowka. Witten's conjecture and property P. *Geom. Topol.* **8** ,295–310, 2004.
33. J. Kollàr. Is there a topological Bogomolov-Miyaoka-Yau inequality? *Pure Appl. Math. Q.* $\mathbf{4}(\mathbf{2})$ Special Issue: In honor of Fedor Bogomolov. Part 1, 203–236, 2008.
34. P. Lisca and G. Matić. Tight contact structures and Seiberg-Witten invariants. *Invent. Math.* $\mathbf{129}(\mathbf{3})$, 509–525, 1997.
35. P. Lisca and A. Stipsicz. Ozváth-Szabó invariants and tight contact 3–manifolds III. *J. Symplectic. Geom.* $\mathbf{5}(\mathbf{4})$, 357–384, 2007.
36. C. Manolescu. Pin(2)-equivariant Seiberg-Witten Floer homology and the triangulation conjecture. *J. Amer. Math. Soc.*, $\mathbf{29}(\mathbf{1})$, 147-176, 2016.
37. T. Mark and B. Tosun. Naturality of Heegaard Floer invariants under positive contact rational surgeries. *J. Diff. Geom.* $\mathbf{110}(\mathbf{2})$, 281–344, 2018.
38. T. Mark and B. Tosun. Obstructing pseudoconvex and contractible Stein fillings for Brieskorn spheres. *Adv. Math.* **335**, 878–895, 2018.
39. T. Mark and B. Tosun. On contact type hypersurfaces in 4-space. Preprint arXiv:2008.02755, 2020.
40. A. Nemethi, and P. Popescu-Pampu. On the Milnor fibres of cyclic quotient singularities. *Proc. London Math. Soc.*, $(\mathbf{3})$, $\mathbf{101}(\mathbf{2})$, 554-588, 2010.
41. S.Nemirovski adn K. Siegel Rationally convex domains and singular Lagrangian surfaces in$\mathbb{C}^2$. *Invent. Math*, **203**, 333-358, 2016.
42. W. Neumann and D. Zagier. A note on an invariant of Fintushel and Stern. Geometry and topology (College Park, Md., 1983/84), Lecture Notes in Math., **1167** 241–244, Springer, Berlin, 1985.

43. W. Neumann and F. Raymond. Seifert manifolds, plumbing, μ-invariant and orientation reversing maps. Algebraic and geometric topology (Proc. Sympos., Univ. California, Santa Barbara, Calif., 1977), Lecture Notes in Math., **664** 163–196, Springer, Berlin, 1978.
44. H. Park, J. Park, D. Shin, and G. Urzúa. Milnor fibers and symplectic fillings of quotient singularities. Preprint, Arxiv:1507.06756. 2015.
45. O. Plamenevskaya and L. Starkston Unexpected Stein fillings, rational surface singularities, and plane curve arrangements. Preprint arXiv:2006.06631, 2020.
46. V. A. Rohlin. New results in the theory of four-dimensional manifolds, *Doklady Akad. Nauk SSSR (N.S.)*, **84** 221–224, 1952.
47. N.Saveliev. Invariants for homology 3-spheres. Encyclopaedia of Mathematical Sciences, vol. 140, Low-Dimensional Topology I, SpringerVerlag, Berlin, 2002.
48. C. H. Taubes. The Seiberg-Witten equations and the Weinstein conjecture, *Geom. Topol.*, **11**, 2117-2202, 2007.
49. C. H. Taubes. Seiberg-Witten and Gromov invariants for symplectic 4–manifolds, Proceedings of the First IP Lecture Series, Volume II,R. Wentworth ed, International Press. Somerville MA 2000.
50. B. Tosun. Tight small Seifert fibered spaces with $e_0 = -2$, *Algebr. Geom. Topol.* $\mathbf{20}(\mathbf{1})$, 1–27, 2020.
51. P. Ozsváth and Z. Szabó. Knot Floer homology and the four-ball genus, *Geom. Topol.* **7**, 615-639, 2003.
52. P. Ozsváth and Z. Szabó. Knot Floer Homology and rational surgeries, *Algebr. Geom. Topol.* **11**, 1-68, 2011.
53. P. Ozsváth and Z. Szabó. Holomorphic disks and three-manifold invariants: properties and applications, *Ann. of Math.* (2) **159**, no. 3, 1159–1245, 2004.
54. P. Ozsváth and Z. Szabó. Heegaard Floer homology and contact structures, *Duke Math. J.* $\mathbf{129}(\mathbf{1})$, 39–61, 2005.
55. P. Ozsváth and Z. Szabó. Holomorphic disks and genus bounds, *Geom. Topol.* **8**, 311–334, 2004.
56. P. Ozsváth and Z. Szabó. Lectures on Heegaard Floer homology, Floer homology, gauge theory, and lowdimensional topology, *Clay Math. Proc., vol. 5, Amer. Math. Soc.*, Providence, RI, 2006, pp. 29–70.
57. K. Yasui. Nonexistence of Stein structures on 4-manifolds and maximal Thurston-Bennequin numbers. *J. Symplectic Geom.* **15** (2017), no. 1, 91–105.

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